On Voronoi Diagrams in the Line Space and Their Generalizations

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***Abstract*—We describe the structure of the Voronoi diagram of lines for a set of points in the plane, thereby making use of an extra dimension. In contrast to previous results in this respect, which were based on the dual representation of the Voronoi diagram under consideration, our approach applies to the primal plane, and generalizes to higher-dimensional spaces.**

***Keywords*-generalized Voronoi diagrams, line space, higher-dimensional spaces**

I. Introduction

A Voronoi diagram is arguably the most widely known and applied geometric structure. Much on its history, properties, and applications can be derived from the survey by Aurenhammer [1] and the book by Okabe et al. [9]. At the same time, a serious effort is now being put on investigating various kinds of *generalized* Voronoi diagrams—see [4] for recent achievements. One of the directions explored when generalizing the concept of Voronoi diagrams is consideration of novel underlying spaces. In the present work, we shall follow this very way.

A two-dimensional *line space* is formed by all the lines in the Euclidean plane. For a set P of point sites in the plane, its Voronoi diagram in the line space is defined as a partition of the latter into Voronoi regions, each corresponding to a distinct site $p \in P$ and consisting of the lines being closer to p than to any other site from P. This kind of Voronoi diagrams was first introduced by Rivière and Schmitt [10]. In particular, they pointed out that such Voronoi diagrams can be easily computed and visualized in dual space (where lines map to points), and subsequently used for processing line localization queries. However, their approach cannot be generalized to higher dimensions. In addition, the structure of the Voronoi diagram in the line space remains of its own interest, though this geometric object obviously cannot be visualized in the plane.

One year later, Rivière [11] introduced and examined Voronoi diagrams of order k in the line space, thereby also exploiting the concept of geometric duality.

Recently, an *onion diagram* was introduced by Bae and Shin [2], being a Voronoi-like structure defined in a parametric space associated with the primal plane. The onion diagram can be used, in particular, for efficiently processing nearest-neighbor line queries for weighted points.

However, in our opinion, it is worth analyzing the primal structure of Voronoi diagrams in a line space as well, for two reasons. First, such investigation may highlight some properties of this Voronoi structure, which are more difficult to observe in the dual plane. Second, the duality-based approaches do not generalize to higher dimensions, while it would be interesting to understand the respective—rather sophisticated—Voronoi structures in higher-dimensional spaces as well.

The goal of this work is to provide a description of the structure of a Voronoi diagram in the primal plane, and to generalize it to three-dimensional case, and further to higher dimensions.

II. Preliminaries

Let $\mathcal{L}$ denote the set of all lines in $\mathbb{R}^2$. Consider a set P of n points in the plane, to which we shall also refer as to *sites*. For any $p \in P$, its Voronoi region $V(p)$ in the Voronoi diagram $\mathrm{Vor}_{\mathcal{L}}(P)$ in $\mathcal{L}$ consists of all the lines in the plane being closer to p than to any other site. Obviously, $\mathrm{Vor}_{\mathcal{L}}(P)$ cannot be visualized in the plane. However, if we consider a duality transform that maps a point $p = (p_x, p_y)$ to a line $p^* : (y = p_x \cdot x + p_y)$, and a line $l : (y = m \cdot x + b)$ to a point $l^* = (m, -b)$, then the dual structure of $\mathrm{Vor}_{\mathcal{L}}(P)$ can be easily visualized (in the dual plane).

Despite the lack of a possibility to "see" $\mathrm{Vor}_{\mathcal{L}}(P)$, it will be useful to understand its structure (in the primal plane)—in particular, because a similar approach to its analysis can be also undertaken in higher dimensions.

III. Two-Dimensional Case

To simplify the exposition, we shall assume that the points from P are in general position (i.e. no three points lie on the same line).

In order to provide a description of $\mathrm{Vor}_{\mathcal{L}}(P)$, we shall need to move from $\mathbb{R}^2$ to a three-dimensional space $\mathcal{SL} = \mathbb{R}^2 \times [0, \pi]$ with a cylindrical topology, meaning that points

$(x, y, 0)$ and (x, y, π) are identified. For any $\phi \in [0, \pi)$, the plane $\mathcal{R}^2_\phi = \mathbb{R}^2 \times \phi$ will contain the subset $\mathcal{L}_\phi$ of lines from $\mathcal{L}$ forming the angle ϕ with the x-axis, and each point $p \in P$ with coordinates (p_x, p_y) will be mapped to an interval $p^{\mathcal{SL}} = (p_x, p_y) \times [0, \pi)$ in $\mathcal{SL}$. For any $\phi \in [0, \pi)$, let $p_\phi = p^{\mathcal{SL}} \cap \mathcal{R}^2_\phi$.

Observe that $p^{\mathcal{SL}}$ is perpendicular to any line under consideration. Consequently, for any $\phi \in [0, \pi)$ and for any line $l_\phi \in \mathcal{L}_\phi$, the distance from l_ϕ to $p^{\mathcal{SL}}$ equals the distance from l_ϕ to p_ϕ.

The Voronoi diagram $\mathrm{Vor}_\mathcal{L}(P)$ gets a natural representation $\mathrm{Vor}_{\mathcal{SL}}(P)$ in $\mathcal{SL}$. For any $\phi \in [0, \pi)$, the intersection $\mathrm{Vor}^\phi_{\mathcal{SL}}(P)$ of $\mathrm{Vor}_{\mathcal{SL}}(P)$ with the plane $\mathcal{R}^2_\phi$ has a fairly simple structure: it represents the Voronoi diagram of the set $P_\phi = \{p_\phi | p \in P\}$ of points in the space of lines composing $\mathcal{L}_\phi$. Unless some two points from P_ϕ lie on the same line from $\mathcal{L}_\phi$, $\mathrm{Vor}^\phi_{\mathcal{SL}}(P)$ is formed of $n-1$ lines from $\mathcal{L}_\phi$ (Fig. 1a,b); in case the line through some two points $p_\phi, q_\phi \in P_\phi$ belongs to $\mathcal{L}_\phi$, $\mathrm{Vor}^\phi_{\mathcal{SL}}(P)$ contains an infinite strip filled by the lines from $\mathcal{L}_\phi$, for which both p_ϕ and q_ϕ are the closest points from P_ϕ (Fig. 1c).

If we examine the structure of $\mathrm{Vor}_{\mathcal{SL}}(P)$ bottom-up (i.e. starting from $\phi = 0$ and increasing it towards π), and attempt to interpret its evolution in terms of $\mathbb{R}^2$, we shall observe the following. Unless some two points from P fall on the same horizontal line, $\mathrm{Vor}^0_{\mathcal{SL}}(P)$ is represented in $\mathbb{R}^2$ by a set of bisectors of consecutive points from P with respect to their vertical order. As we move upwards, each line l being a Voronoi edge rotates in $\mathbb{R}^2$ counterclockwise around the middle point of the segment connecting the two sites, the Voronoi cells of which l bounds, thereby sweeping a two-dimensional face of $\mathrm{Vor}_{\mathcal{SL}}(P)$, until one of those lines happens to pass through some two points $p, q \in P$. Let l_{pq} denote this line. At that very moment when $p, q \in l_{pq}$, a horizontal (i.e. perpendicular to the direction of our movement) infinite strip, which contains l_{pq} and is delimited by the two bisectors on both sides of l_{pq} (if any), is introduced in $\mathrm{Vor}_{\mathcal{SL}}(P)$. Immediately after that, the rotating line l_{pq} starts tracing out the next curved two-dimensional face of $\mathrm{Vor}_{\mathcal{SL}}(P)$, and so on. In case P contains two points lying on the same horizontal line, $\mathrm{Vor}^0_{\mathcal{SL}}(P)$ will contain a horizontal infinite strip composed of horizontal lines, for which those two points are the closest neighbors in P.

Note that in the space $\mathcal{SL}$, the vertical segments representing the points from P will intersect the structure $\mathrm{Vor}_{\mathcal{SL}}(P)$ at the horizontal edges contained inside the infinite horizontal strips (and splitting them into two faces), and corresponding to the lines through the pairs of points from P. In particular, each such vertical segment will intersect $\mathrm{Vor}_{\mathcal{SL}}(P)$ at precisely $(n-1)$ points (see Fig. 2).

We implemented a procedure for visualizing the Voronoi structure $\mathrm{Vor}_{\mathcal{SL}}(P)$ in the space $\mathcal{SL}$; two examples of the respective structures representing a Voronoi diagram $\mathrm{Vor}_\mathcal{L}(P)$ in the line space $\mathcal{L}$ for a set P of 4 points and 6 points, respectively, are provided in Figure 2.

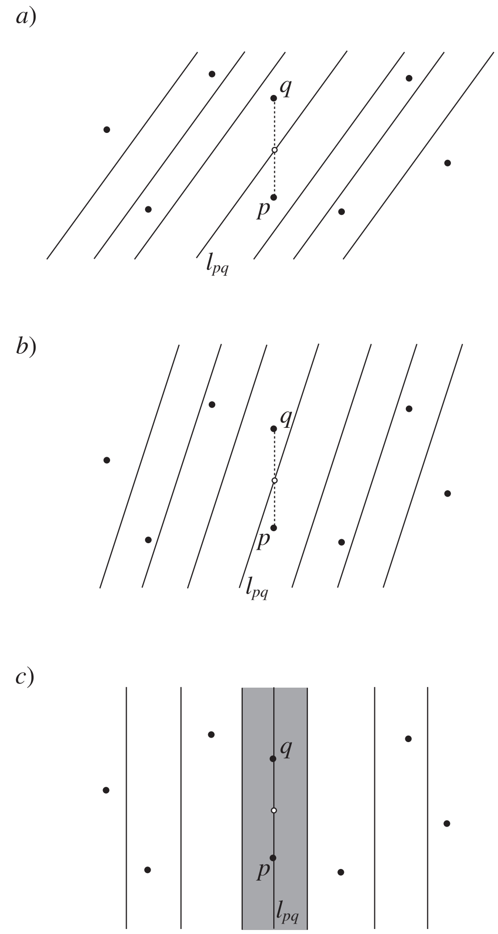

Figure 1. The structure of $\mathrm{Vor}^\phi_{\mathcal{SL}}(P)$ for the given set P of points: a) $\phi = 3\pi/10$, b) $\phi = 2\pi/5$, c) $\phi = \pi/2$. While ϕ increases from $3\pi/10$ to $\pi/2$, the sites p and q remain Voronoi neighbors, and the edge incident to their Voronoi cells is represented by their bisector line l_{pq} in $\mathcal{L}_\phi$. In the plane, l_{pq} rotates around the middle point of the segment pq as ϕ changes from $3\pi/10$ to $\pi/2$. When $\phi = \pi/2$, the line l_{pq} passes through p and q, and the gray vertical infinite strip is composed of the vertical lines, for which p and q are the closest neighbors in P.

IV. Three-Dimensional Case

In the three-dimensional case, we consider the space $\mathcal{H}$ of all planes in $\mathbb{R}^3$, and address the problems of interpreting the Voronoi diagram $\mathrm{Vor}_\mathcal{H}(P)$ of a set P of three-dimensional points in the space $\mathcal{H}$. Again, we assume that the points from P are in general position, meaning that no four points lie in the same plane.

As in the planar case, $\mathrm{Vor}_\mathcal{H}(P)$ can be represented by simpler Voronoi structures, as described below.

Recall that a direction in $\mathbb{R}^3$ can be specified by two angles (Fig. 3). For any plane h in $\mathbb{R}^3$, let us assume that its outer normal $\vec{n}$ is defined by a pair of angles (ϕ, θ), such that $\phi, \theta \in [0, \pi)$. In what follows, when referring to a normal of a plane, we shall mean its outer normal.

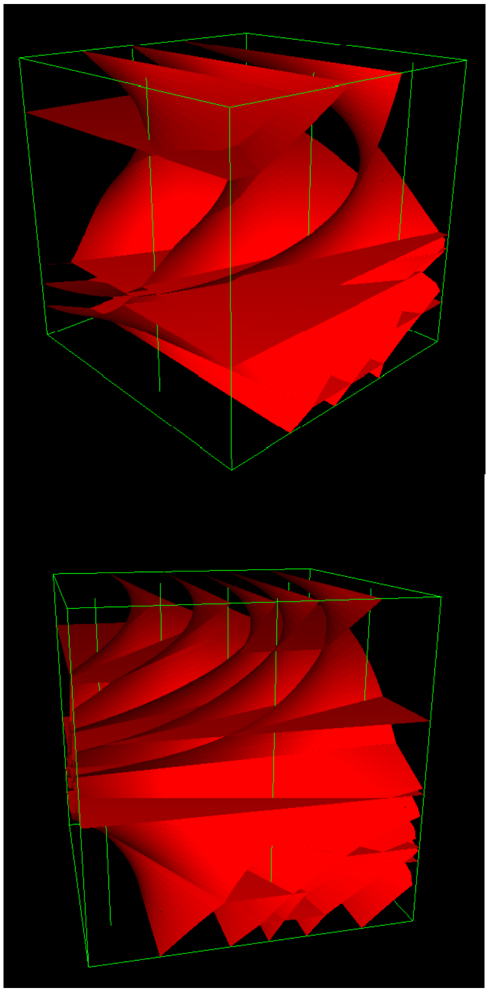

Figure 2. The Voronoi structure $\mathrm{Vor}_{\mathcal{SL}}(P)$ in the space $\mathcal{SL} = \mathbb{R}^2 \times [0, \pi]$ representing the Voronoi diagram $\mathrm{Vor}_{\mathcal{L}}(P)$ in the line space $\mathcal{L}$ for a set P of 4 points (above) and 6 points (below), restricted to the interior of a cube. The points from P are represented in $\mathcal{SL}$ by vertical segments depicted green; to a point $p \in P$, a segment $p^{\mathcal{SL}} = (p_x, p_y) \times [0, \pi)$ corresponds.

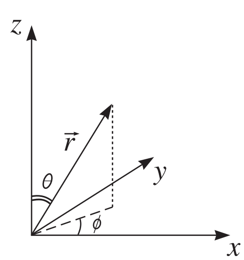

Figure 3. A direction $\vec{r}$ in three-dimensional space can be specified by two angles ϕ and θ.

Let us fix some $\phi, \theta \in [0, \pi)$, and consider a subset $\mathcal{H}_{\phi,\theta}$ of $\mathcal{H}$ consisting of all the planes, the normal of which is defined by the pair (ϕ, θ). Unless some two points from P fall in the same plane from $\mathcal{H}_{\phi,\theta}$, the Voronoi diagram $\mathrm{Vor}^{\phi,\theta}_{\mathcal{H}}(P)$ of P in the subspace $\mathcal{H}_{\phi,\theta}$ is represented by $n-1$ planes, the normal $\vec{n}_{\phi,\theta}$ of each of which is defined by (ϕ, θ), and which pass through the middle points of the segments connecting the consecutive points in a sequence obtained from P by sorting it with respect to the direction $\vec{n}_{\phi,\theta}$. If P does contain two points p and q belonging to the same plane from $\mathcal{H}_{\phi,\theta}$, then $\mathrm{Vor}^{\phi,\theta}_{\mathcal{H}}(P)$ will contain an infinite region consisting of the planes from $\mathcal{H}_{\phi,\theta}$, for which p and q are the closest neighbors in P.

Consequently, $\mathrm{Vor}_{\mathcal{H}}(P)$ can be represented in a five-dimensional space $\mathcal{SH} = \mathbb{R}^3 \times [0, \pi] \times [0, \pi]$, periodic on both "angular" dimensions, in such a way that for any fixed $\phi, \theta \in [0, \pi)$, the intersection of $\mathrm{Vor}_{\mathcal{H}}(P)$ with the three-dimensional subspace of $\mathcal{SH}$ defined by ϕ and θ represents $\mathrm{Vor}^{\phi,\theta}_{\mathcal{H}}(P)$.

V. Higher-Dimensional Case

In the general case, we consider the space $\mathcal{H}$ of all hyperplanes in $\mathbb{R}^d$, and are interested in the structure of the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}(P)$ of a set P of d-dimensional points in the space $\mathcal{H}$. Here we assume that no $d+1$ points from P belong to the same hyperplane.

First, let us recall the formulae defining the generalized polar transform:

$$\begin{aligned} x_1 &= r \cos\phi_1 \\ x_2 &= r \sin\phi_1 \cos\phi_2 \\ x_3 &= r \sin\phi_1 \sin\phi_2 \cos\phi_3 \\ &\dots \\ x_{d-1} &= r \sin\phi_1 \sin\phi_2 \dots \sin\phi_{d-2} \cos\phi_{d-1} \\ x_d &= r \sin\phi_1 \sin\phi_2 \dots \sin\phi_{d-2} \sin\phi_{d-1}. \end{aligned}$$

To a ball $x_1^2 + x_2^2 + \dots + x_d^2 \le R^2$ in the space $x_1 x_2 \dots x_d$, a d-box $0 \le r \le R$, $0 \le \phi_1 \le \pi$, $0 \le \phi_2 \le \pi$, $\dots$, $0 \le \phi_{d-2} \le \pi$, $0 \le \phi_{d-1} \le 2\pi$ in the space $r\phi_1\phi_2 \dots \phi_{d-1}$ corresponds.

A direction in the d-dimensional space can thus be defined by a point on the sphere $x_1^2 + x_2^2 + \dots + x_d^2 = 1$, or, equivalently, by a $(d-1)$-tuple of angles $(\phi_1, \phi_2, \dots, \phi_{d-1})$, where $0 \le \phi_1 \le \pi$, $0 \le \phi_2 \le \pi$, $\dots$, $0 \le \phi_{d-2} \le \pi$, $0 \le \phi_{d-1} \le 2\pi$.

As in the three-dimensional case, we shall decompose $\mathcal{H}$ into sets of hyperplanes having the same outer normal, thereby assuming that the outer normal is the one defined by a $(d-1)$-tuple of angles $(\phi_1, \phi_2, \dots, \phi_{d-1})$, where $\phi_1, \phi_2, \dots, \phi_{d-1} \in [0, \pi)$.

Let us fix some $\phi_1, \phi_2, \dots, \phi_{d-1} \in [0, \pi)$ and consider a subset $\mathcal{H}_{\phi_1,\phi_2,\dots,\phi_{d-1}}$ of $\mathcal{H}$ consisting of all hyperplanes with the outer normal defined by $(\phi_1, \phi_2, \dots, \phi_{d-1})$. As before, unless two points from P lie in the same hyperlnane from $\mathcal{H}_{\phi_1,\phi_2,\dots,\phi_{d-1}}$, the Voronoi diagram $\mathrm{Vor}^{\phi_1,\phi_2,\dots,\phi_{d-1}}_{\mathcal{H}}(P)$ of P in the subspace $\mathcal{H}_{\phi_1,\phi_2,\dots,\phi_{d-1}}$ is represented by $d-1$ hyperplanes with the outer normal $\vec{n}$ defined by $(\phi_1, \phi_2, \dots, \phi_{d-1})$, which pass through the middle points of the segments connecting the neighbor points in a sequence obtained from P by ordering it with respect to the direction $\vec{n}$. In case some two points $p, q \in P$ happen to lie in the same hyperplane from $\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_{d-1}}$, the restricted Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}^{\phi_1,\phi_2,\ldots,\phi_{d-1}}(P)$ will contain an infinite region filled with the hyperplanes from $\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_{d-1}}$, for which p and q are the closest neighbors in P.

We conclude that the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}(P)$ of hyperplanes for a set of points in the d-dimensional space can be represented in a $(2d-1)$-dimensional space $\mathcal{SH} = \mathbb{R}^d \times [0,\pi] \times \cdots \times [0,\pi]$, periodic on each of the $d-1$ "angular" dimensions, so that for any fixed set of angles $\phi_1, \phi_2, \ldots, \phi_{d-1} \in [0,\pi)$, the intersection of $\mathrm{Vor}_{\mathcal{H}}(P)$ with the d-dimensional subspace of $\mathcal{SH}$ defined by $\phi_1, \ldots, \phi_{d-1}$ represents $\mathrm{Vor}_{\mathcal{H}}^{\phi_1,\phi_2,\ldots,\phi_{d-1}}(P)$.

VI. Conclusion

In this work, we have proposed a new way of interpreting the Voronoi diagram of a planar set of points in the line space, which allows to visualize its structure in a three-dimensional space. Unlike the previously used approach to studying such diagrams, based on the concept of duality between points and lines in the plane, our ideas extend to the three-dimensional case, in which a Voronoi diagram of a set of points in the space of all planes is examined, and generalize further to higher dimensions. Though these Voronoi structures are unlikely to allow for a more efficient processing of the nearest neighbor queries than by the existing methods (see [3], [6], [8] for two-dimensional case, [7] for three-dimensional case, and a very recent work [5] for a novel general framework), our results may help to develop a better intuition in regard of their properties, and to admire their inherent beauty.

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