On Voronoi Diagrams in the Planar Line Space and their Generalizations

Dominique Schmitt[1] and Kira Vyatkina[2]

[1] Laboratoire LMIA, Université de Haute-Alsace
4, rue des Frères Lumière, 68093 Mulhouse Cedex, France
Dominique.Schmitt@uha.fr
[2] Algorithmic Biology Laboratory
Saint Petersburg Academic University
8/3 Khlopina str., St Petersburg 194021, Russia
kira@math.spbu.ru

Abstract. We describe the structure of the Voronoi diagram of lines for a set of points in the plane, thereby making use of an extra dimension. In contrast to previous results in this respect, which were based on the dual representation of the Voronoi diagram under consideration, our approach applies to the primal plane. We also generalize it to higher-dimensional hyperplane spaces.

Keywords: generalized Voronoi diagrams, planar line space, higher-dimensional hyperplane spaces

1 Introduction

A Voronoi diagram is arguably the most widely known and applied geometric structure. Much on its history, properties, and applications can be derived from the survey by Aurenhammer [1] and the book by Okabe et al. [10]. At the same time, a serious effort is now being put on investigating various kinds of *generalized* Voronoi diagrams—see [4] for recent achievements. One of the directions explored when generalizing the concept of Voronoi diagrams is consideration of novel underlying spaces. In the present work, we shall follow this very way.

A two-dimensional *line space* is formed by all the lines in the Euclidean plane. For a set P of point sites in the plane, its Voronoi diagram in the line space is defined as a partition of the latter into Voronoi regions, each corresponding to a distinct site $p \in P$ and consisting of the lines being closer to p than to any other site from P. This kind of Voronoi diagrams was first introduced by Rivière and Schmitt [12]. In particular, they pointed out that such Voronoi diagrams can be easily computed and visualized in dual space (where lines map to points), and subsequently used for processing line localization queries.

One year later, Rivière [13] introduced and examined Voronoi diagrams of order k in the line space, thereby also exploiting the concept of geometric duality.

Recently, an *onion diagram* was introduced by Bae and Shin [2], being a Voronoi-like structure defined in a parametric space associated with the primal plane. The onion diagram can be used, in particular, for efficiently processing nearest-neighbor line queries for weighted points.

In our opinion, though the line space Voronoi diagram obviously cannot be visualized in the primal plane, it is worth analyzing its primal structure as well, for two reasons. First, such investigation may highlight some properties of this Voronoi structure, which are more difficult to observe in the dual plane. Second, the duality-based approaches do not generalize to higher-dimensional line spaces, while it would be interesting to understand the respective—rather sophisticated—Voronoi structures in higher-dimensional spaces as well.

The goal of this work is to provide a description of the structure of a Voronoi diagram in the primal plane, making use of an extra dimension. We also give an algorithm to construct and visualize the three-dimensional representation of the planar line space Voronoi diagram. Finally, we show how to extend our method to higher-dimensional hyperplane space Voronoi diagrams.

2 Two-Dimensional Case

Let $\mathcal{L}$ denote the set of all lines in $\mathbb{R}^2$. Consider a set P of n points in the plane, to which we shall also refer as to *sites*. For any $p \in P$, its Voronoi region $V(p)$ in the Voronoi diagram $\mathrm{Vor}_{\mathcal{L}}(P)$ in $\mathcal{L}$ consists of all the lines in the plane being closer to p than to any other site. Obviously, $\mathrm{Vor}_{\mathcal{L}}(P)$ cannot be visualized in the plane. However, if we consider a duality transform that maps a point $p = (p_x, p_y)$ to a line $p^* : (y = p_x \cdot x + p_y)$, and a line $l : (y = m \cdot x + b)$ to a point $l^* = (m, -b)$, then the dual structure of $\mathrm{Vor}_{\mathcal{L}}(P)$ can be easily visualized (in the dual plane).

Despite the lack of a possibility to "see" $\mathrm{Vor}_{\mathcal{L}}(P)$, it will be useful to understand its structure (in the primal plane).

To simplify the exposition, we shall assume that the points from P are in general position (i.e. no three points lie on the same line and no two lines defined by the points from P are parallel).

In order to provide a description of $\mathrm{Vor}_{\mathcal{L}}(P)$, we shall need to move from $\mathbb{R}^2$ to a three-dimensional space $\mathcal{SL} = \mathbb{R}^2 \times [0, \pi]$ with a cylindrical topology, meaning that points $(x, y, 0)$ and (x, y, π) are identified. For any $\phi \in [0, \pi)$, the plane $\mathcal{R}^2_\phi = \mathbb{R}^2 \times \phi$ will contain the subset $\mathcal{L}_\phi$ of lines from $\mathcal{L}$ forming the angle ϕ with the x-axis, and each point $p \in P$ with coordinates (p_x, p_y) will be mapped to an interval $p^{\mathcal{SL}} = (p_x, p_y) \times [0, \pi)$ in $\mathcal{SL}$. For any $\phi \in [0, \pi)$, let $p_\phi = p^{\mathcal{SL}} \cap \mathcal{R}^2_\phi$.

Observe that $p^{\mathcal{SL}}$ is perpendicular to any line under consideration. Consequently, for any $\phi \in [0, \pi)$ and for any line $l_\phi \in \mathcal{L}_\phi$, the distance from l_ϕ to $p^{\mathcal{SL}}$ equals the distance from l_ϕ to p_ϕ.

The Voronoi diagram $\mathrm{Vor}_{\mathcal{L}}(P)$ gets a natural representation $\mathrm{Vor}_{\mathcal{SL}}(P)$ in $\mathcal{SL}$. For any $\phi \in [0, \pi)$, the intersection $\mathrm{Vor}^\phi_{\mathcal{SL}}(P)$ of $\mathrm{Vor}_{\mathcal{SL}}(P)$ with the plane $\mathcal{R}^2_\phi$ has a fairly simple structure: it represents the Voronoi diagram of the set $P_\phi = \{p_\phi | p \in P\}$ of points in the space of lines composing $\mathcal{L}_\phi$. Unless some two points from P_ϕ lie on the same line from $\mathcal{L}_\phi$, $\mathrm{Vor}^\phi_{\mathcal{SL}}(P)$ is formed of $n-1$ lines from $\mathcal{L}_\phi$ (Fig. 1a,b,d); in case the line through some two points $p_\phi, q_\phi \in P_\phi$ belongs to $\mathcal{L}_\phi$,

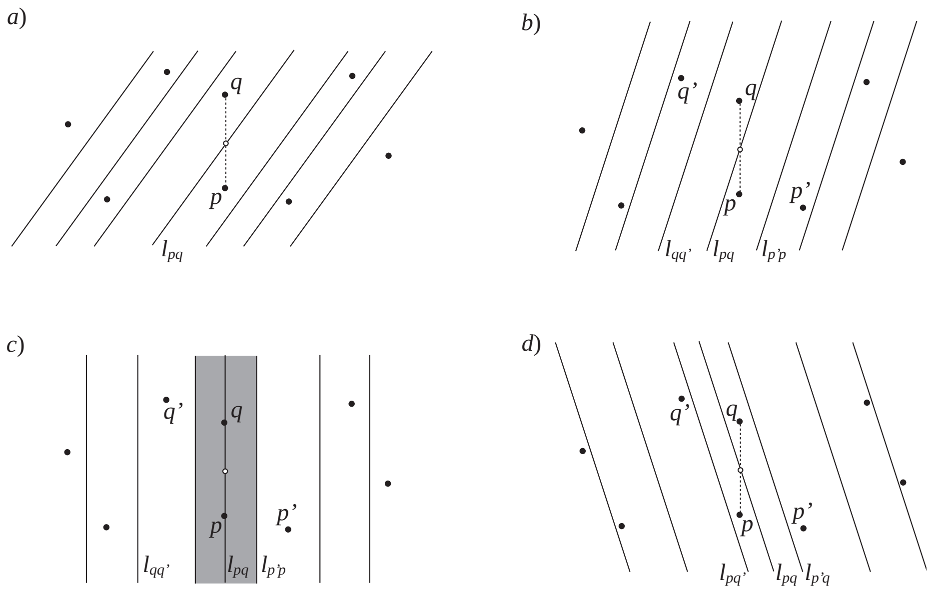

Fig. 1. The structure of $\mathrm{Vor}^{\phi}_{\mathcal{SL}}(P)$ for the given set P of points: a) $\phi = 3\pi/10$, b) $\phi = 2\pi/5$, c) $\phi = \pi/2$, d) $\phi = 3\pi/5$. While ϕ increases from $3\pi/10$ to $3\pi/5$, the sites p and q remain Voronoi neighbors, and the edge incident to their Voronoi cells is represented by their bisector line l_{pq} in $\mathcal{L}_{\phi}$. In the plane, l_{pq} rotates around the middle point of the segment pq as ϕ changes from $3\pi/10$ to $3\pi/5$. When $\phi = \pi/2$, the line l_{pq} passes through p and q, and the gray vertical infinite strip is composed of the vertical lines, for which p and q are the closest neighbors in P.

$\mathrm{Vor}^{\phi}_{\mathcal{SL}}(P)$ contains an infinite strip filled by the lines from $\mathcal{L}_{\phi}$, for which both p_{ϕ} and q_{ϕ} are the closest points from P_{ϕ} (Fig. 1c).

If we examine the structure of $\mathrm{Vor}_{\mathcal{SL}}(P)$ bottom-up (i.e. starting from $\phi = 0$ and increasing it towards π), and attempt to interpret its evolution in terms of $\mathbb{R}^2$, we shall observe the following. Unless some two points from P fall on the same horizontal line, $\mathrm{Vor}^{0}_{\mathcal{SL}}(P)$ is represented in $\mathbb{R}^2$ by a set of (horizontal) bisectors of consecutive points from P with respect to their vertical order. As we move upwards, each line l being a Voronoi edge rotates in $\mathbb{R}^2$ counterclockwise around the middle point of the segment connecting the two sites, the Voronoi cells of which l bounds, thereby sweeping a two-dimensional face of $\mathrm{Vor}_{\mathcal{SL}}(P)$, until one of those lines happens to pass through some two points $p, q \in P$. Let l_{pq} denote this line, and let $l_{p'p}$ (resp. $l_{qq'}$) be the other line bounding the Voronoi cell of p (resp. of q), if this line exists (see Fig. 1b). Clearly, one of $l_{p'p}$ and $l_{qq'}$ does not exist if and only if pq is an edge of the convex hull $conv(P)$ of P. None of them exists if $n = 2$. At that very moment when $p, q \in l_{pq}$, two horizontal (i.e. perpendicular to the direction of our movement) faces, each being an infinite strip bounded on one side by l_{pq}, are introduced in $\mathrm{Vor}_{\mathcal{SL}}(P)$; thus, l_{pq} represents their common edge. Together those two faces form an infinite strip containing l_{pq} inside. If $l_{p'p}$ exists, one face is delimited by the line $l_{p'p}$, which thus introduces another edge in $\mathrm{Vor}_{\mathcal{SL}}(P)$ (see Fig. 1c). Otherwise, the respective face is unbounded on one side of l_{pq}. The same happens on the other side of l_{pq}, depending on the fact whether $l_{qq'}$ exists or not. Hence, one, two, or three edges are introduced in $\mathrm{Vor}_{\mathcal{SL}}(P)$ when $p, q \in l_{pq}$. Immediately after that, the rotating line l_{pq} starts tracing out the next curved two-dimensional face of $\mathrm{Vor}_{\mathcal{SL}}(P)$. The line $l_{p'p}$ (if it exists) is replaced by the line $l_{p'q}$ that passes through the midpoint of $p'q$ and will trace out a new face of $\mathrm{Vor}_{\mathcal{SL}}(P)$, and at this moment, coincides with $l_{p'p}$ (see Fig. 1d). In the same way, $l_{qq'}$ is replaced by $l_{pq'}$. Every other line l being a Voronoi edge in $\mathbb{R}^2$, continues tracing out a same face of $\mathrm{Vor}_{\mathcal{SL}}(P)$, until some line hits some two points of P. And so on. In case P contains two points lying on the same horizontal line, $\mathrm{Vor}^0_{\mathcal{SL}}(P)$ will contain a horizontal infinite strip composed of two horizontal faces sharing an edge.

Note that in the space $\mathcal{SL}$, the vertical segment representing a point $p \in P$ will intersect the structure $\mathrm{Vor}_{\mathcal{SL}}(P)$ at the horizontal edges corresponding to the lines through p and each of the points $q \in P \setminus \{p\}$ (and namely, the edges contained inside the infinite horizontal strips and splitting those into two faces). Hence, each such vertical segment will intersect $\mathrm{Vor}_{\mathcal{SL}}(P)$ at precisely $n-1$ points (see Fig. 3).

Moreover, observe that if we direct the sweeping lines in such a way that their angles ϕ with the x-axis belong to $[0, \pi]$, then at each moment of the sweeping, we can sort from right to left the $n-1$ lines sweeping the curved faces of $\mathrm{Vor}_{\mathcal{SL}}(P)$. Throughout the process, the lines that occur at the i^{th} place together sweep a connected set of faces of $\mathrm{Vor}_{\mathcal{SL}}(P)$, thus producing a ruled surface, which is piecewise helicoidal with vertical axis. The $n-1$ surfaces defined that way are pairwise disjoint. To retrieve the topological structure of the diagram, we shall need, in particular, to identify the i^{th} line for $\phi = 0$ with the $(n-i)^{th}$ line for $\phi = \pi$, thus forming $\lceil n/2 \rceil$ annuli.

Now, it is easy to compute the size of $\mathrm{Vor}_{\mathcal{SL}}(P)$, using the above description. For every ϕ being an angle between a line through some two points $p, q \in P$ and the x-axis, the plane $\mathcal{R}^2_\phi$ contains exactly two horizontal faces of $\mathrm{Vor}_{\mathcal{SL}}(P)$. Since all horizontal faces are contained in those planes, there are $n(n-1)$ such faces. Each pair of faces is delimited by two or three edges of $\mathrm{Vor}_{\mathcal{SL}}(P)$ depending on whether pq is an edge of $conv(P)$ or not (here we assume $n > 2$). Since all the edges are defined that way, $\mathrm{Vor}_{\mathcal{SL}}(P)$ admits $3(n(n-1))/2 - n'$ edges, where n' is the number of vertices of $conv(P)$. These edges split the annular ruled surfaces into the helicoidal faces of $\mathrm{Vor}_{\mathcal{SL}}(P)$, thus defining $3(n(n-1))/2 - n'$ such faces. Finally, we note that the $n(n-1)$ horizontal faces are the "floors" and the "ceilings" of the $n(n-1)$ regions of $\mathrm{Vor}_{\mathcal{SL}}(P)$. Thus, the total size of $\mathrm{Vor}_{\mathcal{SL}}(P)$ is in $\Theta(n^2)$.

3 Algorithm in Two Dimensions

A natural way to construct $\mathrm{Vor}_{\mathcal{SL}}(P)$ consists in simulating the sweeping of the plane by a set of oriented parallel lines, by varying the angle ϕ they make with the x-axis from 0 to π (π left out). The events of the sweeping are the $n(n-1)/2$ moments when a sweep-line hits two points of P. All the respective ordered pairs of points (p, q) are first placed in the positions $[1, \ldots, n(n-1)/2]$ of an event-array E (here, p and q are ordered in such a way that the straight line (pq) is oriented in the same direction as the sweep-lines). The array E is then sorted by increasing ϕ angles, resulting in an $O(n^2 \log n)$ initialization step.

The algorithm also needs three additional arrays:

- an array $T[1, \ldots, n]$ that contains at each moment ϕ of the sweeping, the points of P in the order in which they are encountered by an oriented sweep-line with angle ϕ moving from right to left. For $\phi = 0$, the points in T are sorted by increasing y-coordinates. The position of every point of P in T is also maintained.
- two arrays $first[1, \ldots, n-1]$ and $last[1, \ldots, n-1]$ whose positions i contain respectively the first and the last edge of $\mathrm{Vor}_{\mathcal{SL}}(P)$ already created on the i^{th} ruled surface (i.e., the ruled surface swept by the line that constantly "separates" the cells of the i^{th} and $(i{+}1)^{th}$ points in T). Initially, all positions of these arrays are $NULL$.

Once the four arrays are initialized, the following algorithm then constructs $\mathrm{Vor}_{\mathcal{SL}}(P)$:

For every $e \in [1, \ldots, n(n-1)/2]$
  let $(p, q) \longleftarrow E[e]$
  let i be the current position of p in T //here, q is necessarily in $T[i+1]$

  if $last[i] = NULL$
    initialize $first[i]$ and $last[i]$ with the line (pq)
  else
    add a face to $\mathrm{Vor}_{\mathcal{SL}}(P)$ between the lines $last[i]$ and (pq), which is a helicoidal surface with axis the vertical through the midpoint of pq
    $last[i] \longleftarrow (pq)$
  endif

  if $i = 1$ // pq is an edge of $conv(P)$
    add a face to $\mathrm{Vor}_{\mathcal{SL}}(P)$, which is the horizontal half-plane on the right of (pq)
  else
    let $p' \longleftarrow T[i-1]$
    let $l_{p'p}$ be the line parallel to (pq) passing through the midpoint of $p'p$
    if $last[i-1] = NULL$
      initialize $first[i-1]$ and $last[i-1]$ with the line $l_{p'p}$
    else

add a face to $\text{Vor}_{\mathcal{SL}}(P)$ between the lines $last[i-1]$ and $l_{p'p}$, which is a helicoidal surface with axis the vertical through the midpoint of $p'p$

$last[i-1] \longleftarrow l_{p'p}$

endif

add a face to $\text{Vor}_{\mathcal{SL}}(P)$, which is the horizontal strip delimited by the lines (pq) and $l_{p'p}$

endif

if $i+1 = n$ // pq is an edge of $conv(P)$

add a face to $\text{Vor}_{\mathcal{SL}}(P)$, which is the horizontal half-plane on the left of (pq)

else

let $q' \longleftarrow T[i+2]$

let $l_{qq'}$ be the line parallel to (pq) passing through the midpoint of qq'

if $last[i+1] = NULL$

initialize $first[i+1]$ and $last[i+1]$ with the line $l_{qq'}$

else

add a face to $\text{Vor}_{\mathcal{SL}}(P)$ between the lines $last[i+1]$ and $l_{qq'}$, which is a helicoidal surface with axis the vertical through the midpoint of qq'

$last[i+1] \longleftarrow l_{qq'}$

endif

add a face to $\text{Vor}_{\mathcal{SL}}(P)$, which is the horizontal strip delimited by the lines (pq) and $l_{qq'}$

endif

swap p and q in T

done

For every $i \in [1, \ldots, n-1]$

add a helicoidal face to $\text{Vor}_{\mathcal{SL}}(P)$ between the lines $last[i]$ and $first[n-i]$

done

Obviously, the complexity of this algorithm is in $O(n^2)$, that is, linear in the size of $\text{Vor}_{\mathcal{SL}}(P)$. Hence, the overall complexity of the construction of $\text{Vor}_{\mathcal{SL}}(P)$ is dominated by the sorting of the event-array E.

In [12], it has been shown that, in dual space, the line space Voronoi diagram of a set of points P can be constructed in $O(n^2)$ time, by first computing the arrangement of the dual lines of P. Clearly, this algorithm can be adapted to construct $\text{Vor}_{\mathcal{SL}}(P)$ in $O(n^2)$ time. However, if we want to construct the line space Voronoi diagram in $O(n^2)$ time without having to compute the dual line arrangement, we have to use topological sweeping methods like those implemented to sweep line arrangements [5]. Consider an ordered pair (p, q) of points that are consecutive in T at a moment ϕ of the sweeping in our algorithm. Let p' and q' be the points that respectively precedes and follows p and q in T (the case when one of these points does not exist can be treated in a similar way). Suppose that, while sweeping by increasing angles, the four points p', p, q, and q' remain consecutive in this order in T until the moment ϕ' when a sweep line passes through p and q. Clearly, the pair (p,q) can be treated by our algorithm at any moment between ϕ and ϕ', even if there exist pairs (s,t) making an angle with the x-axis between ϕ and ϕ'. This shows that a topological sweep applies to our algorithm. However, the usual topological sweeps only respect the following constraint: if two pairs of points share a common point then the one defining the line making the smallest angle with the x-axis must be treated first [11]. This constraint is not sufficient for our algorithm, as shown in Fig. 2. Hence, the problem of generating a sweeping order in $O(n^2)$ time to construct the line space Voronoi diagram without calculating first the dual line arrangement remains open.

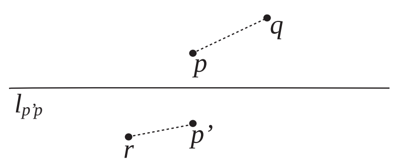

Fig. 2. Let ϕ and ϕ' be the respective angles that the lines (rp') and (pq) make with the x-axis. In $\mathrm{Vor}_{\mathcal{SL}}(P)$, the ruled surface that contains $l_{p'p}$ is composed, between the planes $\mathcal{R}_0^2$ and $\mathcal{R}_\phi^2$, by a helicoidal face with axis the vertical through the midpoint of $p'p$ and, between the planes $\mathcal{R}_\phi^2$ and $\mathcal{R}_{\phi'}^2$, by a helicoidal face with axis the vertical through the midpoint of rp. If our algorithm treats the pair (p,q) before (r,p'), it constructs a single helicoidal face containing $l_{p'p}$ between $\mathcal{R}_0^2$ and $\mathcal{R}_{\phi'}^2$.

We implemented a procedure for visualizing the Voronoi structure $\mathrm{Vor}_{\mathcal{SL}}(P)$ in the space $\mathcal{SL}$; two examples of the respective structures representing a Voronoi diagram $\mathrm{Vor}_{\mathcal{L}}(P)$ in the line space $\mathcal{L}$ for a set P of 4 points and 6 points, respectively, are provided in Fig. 3.

4 Higher-Dimensional Case

In the three-dimensional case, we consider the space $\mathcal{H}$ of all planes in $\mathbb{R}^3$, and address the problems of interpreting the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}(P)$ of a set P of three-dimensional points in the space $\mathcal{H}$. Again, we assume that the points from P are in general position, meaning that no three points lie in the same plane.

As in the planar case, $\mathrm{Vor}_{\mathcal{H}}(P)$ can be represented by simpler Voronoi structures, as described below.

Recall that a direction in $\mathbb{R}^3$ can be specified by two angles (Fig. 4). For any plane h in $\mathbb{R}^3$, let us assume that its outer normal $\boldsymbol{n}$ is defined by a pair of angles (ϕ,θ), such that $\phi,\theta \in [0,\pi)$. In what follows, when referring to a normal of a plane, we shall mean its outer normal.

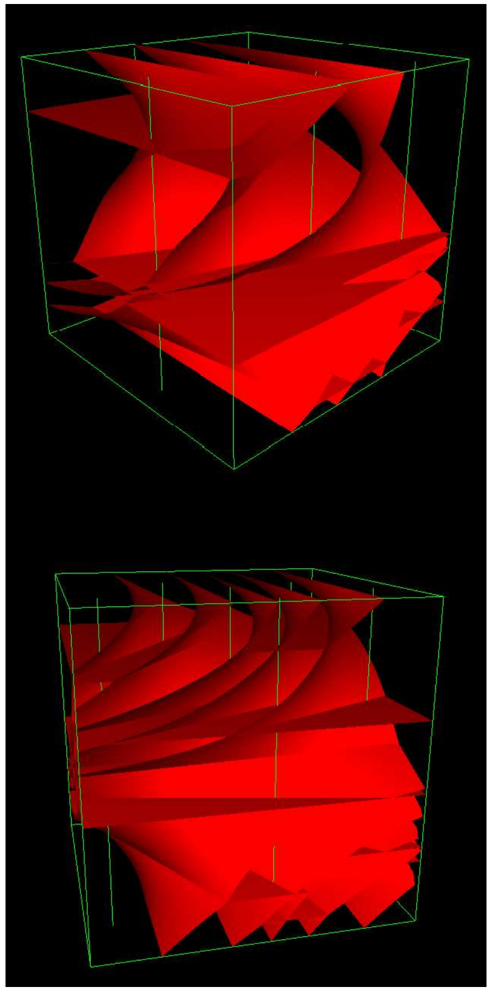

Fig. 3. The Voronoi structure $\mathrm{Vor}_{\mathcal{SL}}(P)$ in the space $\mathcal{SL} = \mathbb{R}^2 \times [0, \pi]$ representing the Voronoi diagram $\mathrm{Vor}_{\mathcal{L}}(P)$ in the line space $\mathcal{L}$ for a set P of 4 points (above) and 6 points (below), restricted to the interior of a cube. The points from P are represented in $\mathcal{SL}$ by vertical segments depicted green; to a point $p \in P$, a segment $p^{\mathcal{SL}} = (p_x, p_y) \times [0, \pi)$ corresponds.

Let us fix some $\phi, \theta \in [0, \pi)$, and consider a subset $\mathcal{H}_{\phi,\theta}$ of $\mathcal{H}$ consisting of all the planes, the normal of which is defined by the pair (ϕ, θ). Unless some two points from P fall in the same plane from $\mathcal{H}_{\phi,\theta}$, the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}^{\phi,\theta}(P)$ of P in the subspace $\mathcal{H}_{\phi,\theta}$ is represented by $n-1$ planes, the normal $\boldsymbol{n}_{\phi,\theta}$ of each of which is defined by (ϕ, θ), and which pass through the middle points of the segments connecting the consecutive points in a sequence obtained from P by sorting it with respect to the direction $\boldsymbol{n}_{\phi,\theta}$. If P does contain two points

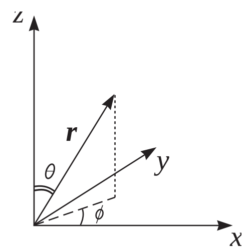

Fig. 4. A direction $\boldsymbol{r}$ in three-dimensional space can be specified by two angles ϕ and θ.

p and q belonging to the same plane from $\mathcal{H}_{\phi,\theta}$, then $\mathrm{Vor}_{\mathcal{H}}^{\phi,\theta}(P)$ will contain an infinite region consisting of the planes from $\mathcal{H}_{\phi,\theta}$, for which p and q are the closest neighbors in P.

Consequently, $\mathrm{Vor}_{\mathcal{H}}(P)$ can be represented in a five-dimensional space $\mathcal{SH} = \mathbb{R}^3 \times [0,\pi] \times [0,\pi]$, periodic on both "angular" dimensions, in such a way that for any fixed $\phi, \theta \in [0,\pi)$, the intersection of $\mathrm{Vor}_{\mathcal{H}}(P)$ with the three-dimensional subspace of $\mathcal{SH}$ defined by ϕ and θ represents $\mathrm{Vor}_{\mathcal{H}}^{\phi,\theta}(P)$.

In the general case, we consider the space $\mathcal{H}$ of all hyperplanes in $\mathbb{R}^d$, and are interested in the structure of the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}(P)$ of a set P of d-dimensional points in the space $\mathcal{H}$. Here we assume that no d points from P belong to the same hyperplane.

First, let us recall the formulae defining the generalized polar transform:

$$
\begin{aligned}
x_1 &= r\cos\phi_1 \\
x_2 &= r\sin\phi_1\cos\phi_2 \\
x_3 &= r\sin\phi_1\sin\phi_2\cos\phi_3 \\
&\dots \\
x_{d-1} &= r\sin\phi_1\sin\phi_2\dots\sin\phi_{d-2}\cos\phi_{d-1} \\
x_d &= r\sin\phi_1\sin\phi_2\dots\sin\phi_{d-2}\sin\phi_{d-1}.
\end{aligned}
$$

To a ball $x_1^2 + x_2^2 + \cdots + x_d^2 \le R^2$ in the space $x_1x_2\dots x_d$, a d-box $0 \le r \le R$, $0 \le \phi_1 \le \pi$, $0 \le \phi_2 \le \pi$, $\dots$, $0 \le \phi_{d-2} \le \pi$, $0 \le \phi_{d-1} \le 2\pi$ in the space $r\phi_1\phi_2\dots\phi_{d-1}$ corresponds.

A direction in the d-dimensional space can thus be defined by a point on the sphere $x_1^2 + x_2^2 + \cdots + x_d^2 = 1$, or, equivalently, by a $(d-1)$-tuple of angles $(\phi_1, \phi_2, \dots, \phi_{d-1})$, where $0 \le \phi_1 \le \pi$, $0 \le \phi_2 \le \pi$, $\dots$, $0 \le \phi_{d-2} \le \pi$, $0 \le \phi_{d-1} \le 2\pi$.

As in the three-dimensional case, we shall decompose $\mathcal{H}$ into sets of hyperplanes having the same outer normal, thereby assuming that the outer normal is the one defined by a $(d-1)$-tuple of angles $(\phi_1, \phi_2, \dots, \phi_{d-1})$, where $\phi_1, \phi_2, \dots, \phi_{d-1} \in [0,\pi)$.

Let us fix some $\phi_1, \phi_2, \dots, \phi_{d-1} \in [0,\pi)$ and consider a subset $\mathcal{H}_{\phi_1,\phi_2,\dots,\phi_{d-1}}$ of $\mathcal{H}$ consisting of all hyperplanes with the outer normal defined by the $(d-1)$-

tuple $(\phi_1, \phi_2, \ldots, \phi_{d-1})$. As before, unless two points from P lie in the same hyperplane from $\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_{d-1}}$, the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}^{\phi_1,\phi_2,\ldots,\phi_{d-1}}(P)$ of P in the subspace $\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_{d-1}}$ is represented by $d-1$ hyperplanes with the outer normal $\boldsymbol{n}$ defined by $(\phi_1, \phi_2, \ldots, \phi_{d-1})$, which pass through the middle points of the segments connecting the neighbor points in a sequence obtained from P by ordering it with respect to the direction $\boldsymbol{n}$. In case some two points $p, q \in P$ happen to lie in the same hyperplane from $\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_{d-1}}$, the restricted Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}^{\phi_1,\phi_2,\ldots,\phi_{d-1}}(P)$ will contain an infinite region filled with the hyperplanes from $\mathcal{H}_{\phi_1,\phi_2,\ldots,\phi_{d-1}}$, for which p and q are the closest neighbors in P.

We conclude that the Voronoi diagram $\mathrm{Vor}_{\mathcal{H}}(P)$ of hyperplanes for a set of points in the d-dimensional space can be represented in a $(2d-1)$-dimensional space $\mathcal{SH} = \mathbb{R}^d \times [0, \pi] \times \cdots \times [0, \pi]$, periodic on each of the $d-1$ "angular" dimensions, so that for any fixed set of angles $\phi_1, \phi_2, \ldots, \phi_{d-1} \in [0, \pi)$, the intersection of $\mathrm{Vor}_{\mathcal{H}}(P)$ with the d-dimensional subspace of $\mathcal{SH}$ defined by ϕ_1, $\ldots$, ϕ_{d-1} represents $\mathrm{Vor}_{\mathcal{H}}^{\phi_1,\phi_2,\ldots,\phi_{d-1}}(P)$.

5 Conclusion

In this work, we have proposed a new way of interpreting the Voronoi diagram of a planar set of points in the line space, which allows to visualize its structure in a three-dimensional space. Our ideas extend to the three-dimensional case, in which a Voronoi diagram of a set of points in the space of all planes is examined, and generalize further to higher dimensions. Though these Voronoi structures are unlikely to allow for a more efficient processing of the nearest neighbor queries than by the existing methods (see [3, 7, 9] for two-dimensional case, [8] for three-dimensional case, and a very recent work [6] for a novel general framework), our results may help to develop a better intuition in regard of their properties, and to admire their inherent beauty. We also hope that our approach will help to study Voronoi diagrams in higher-dimensional line spaces (instead of hyperplane spaces), where the methods based on the concept of duality do not work.

References

1. F. Aurenhammer, *Voronoi diagram—A survey of a fundamental geometric data structure.* ACM Computing Surveys, 23 (3), 345–405, 1991.
2. S. W. Bae and C.-S. Shin, *The Onion Diagram: A Voronoi-Like Tessellation of a Planar Line Space and Its Applications.* In: O. Cheong, K.-Y. Chwa, and K. Park (Eds.), Proc. 21st Int. Symp. on Algorithms and Computation (ISAAC'2010), Part II, LNCS 6507, Springer, 230–241, 2010.
3. R. Cole and C. Yap, Geometric retrieval problems. In: Proc 24th IEEE Symp. Foundation of Computer Science (FOCS), 112–121, 1983.
4. M. Gavrilova (Ed.), *Generalized Voronoi Diagram: A Geometry-Based Approach to Computational Intelligence.* Springer, 2008.
5. H. Edelsbrunner and L. J. Guibas, *Topologically sweeping an arrangement.* J. Comput. Syst. Sci., 38, 165–194, 1989. Corrigendum in 42, 249–251, 1991.

6. T. Hruz and M. Schoengens, *A Simple Framework for the Generalized Nearest Neighbor Problem*. In: Proc. 13th Scandinavian Symp. and Workshops on Algorithm Theory (SWAT 2012), LNCS 7357, Springer, 8394, 2012.
7. D. T. Lee and Y. Chiang, *The power of geometric duality revisited*. Inf. Process. Lett., 21, 117–122, 1985.
8. P. Mitra and A. Mukhopadhyay, *Computing a Closest Point to a Query Hyperplane in Three and Higher Dimensions*. In: Proc. 3rd Int. Conf. Comput. Sci. and its Appl. (ICCSA 2003), LNCS 2669, Springer, 787–796, 2003.
9. S. C. Nandy, S. Das and P. P. Goswami, An efficient k nearest neighbors searching algorithm for a query line. Theor. Comp. Sci., 299, 273-288, 2003.
10. A. Okabe, A. Boots, B. Sugihara, and S.N. Chui, Spatial Tessellations, 2nd ed., Wiley, 2000.
11. M.H. Overmars and E. Welzl, *New methods for computing visibility graphs*. In: Proc. 4th Annu. Symp. Comput. Geom. (ACM SoCG), ACM Press, 164–171, 1988.
12. S. Rivière and D. Schmitt, *Two-dimensional line space Voronoi Diagram*. In: Proc. 4th Int. Symp. on Voronoi Diagrams in Science and Engineering (ISVD 2007), IEEE CS, 168–175, 2007.
13. S. Rivière, *Two-dimensional order-k line space Voronoi diagram*. In: Voronoi's Impact on Modern Science, Book 4, Vol. 2, Proc. 5th Int. Symp. on Voronoi Diagrams in Science and Engineering (ISVD 2008), 192-202, 2008.