Order-k Triangulations of Convex Inclusion Chains in the Plane

Wael El Oraiby* Dominique Schmitt*

Abstract

Given a set V of n points in the plane, we show that there is a strong connexion between the k-sets of a convex inclusion chain of V introduced in [5] and the centroid triangulations of V defined in [8]. We also show that one of these triangulations can be constructed in $O(n \log n + k(n-k) \log^2 k)$ time.

1 Introduction

Given a finite set V of n points in the Euclidean plane (no three of them being collinear) and an integer k ($0 < k \leq n$), the k-sets of V are the subsets of k points of V that can be strictly separated from the rest by a straight line. The numbers of k-sets have been studied in various ways in computational and combinatorial geometry (see [4], [12], and [10] for some best bounds known in the plane). In [5], we have given a new invariant of the number of k-sets, in connexion with convex inclusion chains of V. Such a chain is an ordering $\mathcal{V} = (v_1, v_2, ..., v_n)$ of the points of V such that, for every $i \in \{2, ..., n\}$, v_i does not belong to the convex hull $conv(S_{i-1})$ (with $S_i = \{v_1, ..., v_i\}$, for all $i \in \{1, ..., n\}$). The set of k-sets of the convex inclusion chain $\mathcal{V}$ is then the set of distinct k-sets of $S_k, S_{k+1}, ..., S_n$. We have shown that the number of these k-sets does not depend on the chosen chain and, surprisingly, it is equal to the number of regions of the order-k Voronoi diagram of V.

Independently, while studying multivariate splines, Lyu and Snoeyink have introduced the notion of centroid triangulation [8]. It is a generalization of the order-k Delaunay diagram, which is dual to the order-k Voronoi diagram [3, 11] (note that this order-k Delaunay diagram has nothing to do with the order-k Delaunay triangulation of [6]). For $k \leq 3$, Lyu and Snoeyink have proven the correctness of there constructive definition of centroid triangulations and they have conjectured that it also holds for $k > 3$.

In this paper we establish the relation between the k-sets of the convex inclusion chains of a point set V and the centroid triangulations of V. More precisely, we show that, for all k, the centroids of the k-sets of a convex inclusion chain of V are the vertices of a centroid triangulation of V. We call this triangulation the order-k triangulation of the convex inclusion chain.

*Laboratoire MIA, Université de Haute-Alsace, Mulhouse, France {Wael.El-Oraiby, Dominique.Schmitt}@uha.fr

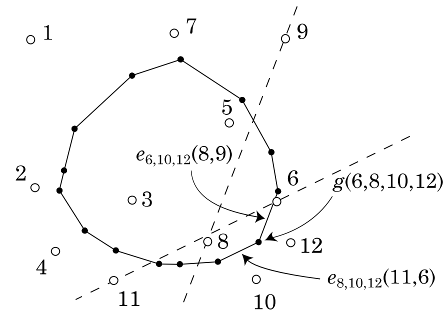

Figure 1: Edges and vertices of a 4-set polygon of 12 points.

On the one hand, this result allows us to find, for all k, a family of centroid triangulations that verify the definition of Lyu and Snoeyink. On the other hand, it is a first step toward the understanding why the number of k-sets of a convex inclusion chain is equal to the number of regions of the order-k Voronoi diagram.

Finally, we show that a particular centroid triangulation can be constructed in $O(n \log n+k(n-k) \log^2 k)$ time. This improves the algorithm that follows from the constructive definition of Lyu and Snoeyink whose time complexity is at least $O(n \log n + k^2(n-k))$.

2 k-set polygons

Given two points s and t of V, we denote by st the closed line segment with endpoints s and t oriented from s to t, by (st) the oriented straight line generated by st, and by $(st)^-$ the open half plane on the right of (st). For any subset E of the plane, we denote by $\overline{E}$ the closure of E.

Let $g^k(V)$ be the k-set polygon of V, i.e., the convex hull of the centroids of all the k-element subsets of V. Notice that, $g^1(V)$ is the convex hull $conv(V)$ of V and $g^n(V)$ is a unique point, the centroid $g(V)$ of V.

We first recall two important properties of the vertices and edges of $g^k(V)$ given by Andrzejak and Fukuda [1], and by Andrzejak and Welzl [2] (see Figure 1 for an illustration).

Proposition 1 *(i) $g(T)$ is a vertex of $g^k(V)$ if and only if T is a k-set of V.*

(ii) $g(T)g(T')$ is a counterclockwise oriented edge of $g^k(V)$ if and only if there exist two points s and t

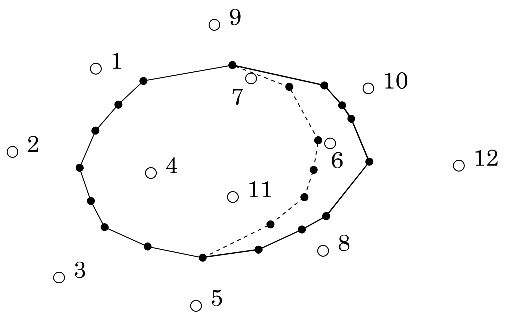

Figure 2: Construction of the 4-set polygon of $S \cup \{12\}$ from the 4-set polygon of $S = \{1, ..., 11\}$. The edges to remove are in dashed lines and the edges to create in bold lines.

of V and a subset P of $k-1$ points of V such that $T = P \cup \{s\}$, $T' = P \cup \{t\}$, and $V \cap (st)^- = P$.

From now on, any such oriented edge will be denoted by $e_P(s,t)$. Obviously, $e_P(s,t)$ is parallel to (st) and it is not difficult to see that the any line that separates the vertex $g(T)$ from the vertices of $g^k(V)$ is parallel to a line that separates T from V (and conversely).

Let S be a non empty subset of V and v be a point of $V \setminus conv(S)$, and consider the edges to remove and the ones to create when constructing $g^k(S \cup \{v\})$ from $g^k(S)$ (see Figure 2). From [5], we know that:

Proposition 2 *(i) The edges to remove are the edges $e_P(s,t)$ of $g^k(S)$ with $v \in (st)^-$. Together with their endpoints, they form a connected polygonal line $\mathcal{D}^k_{S,v}$.*

(ii) The edges to create form a connected polygonal line of at least two edges. The first (resp. last) of them (in counterclockwise direction) is of the form $e_P(s,t)$ with $t = v$ (resp. $s = v$). The other edges to create form a polygonal line $\mathcal{C}^k_{S,v}$ of edges of the form $e_P(s,t)$ with $v \in P$.

For every vertex $g(T_i)$ of $\mathcal{D}^k_{S,v}$, let $\varphi(T_i)$ be the set:

- of vertices $g(T)$ of $\mathcal{C}^k_{S,v}$ such that T and T_i can be separated respectively from $S \cup \{v\}$ and from S by two parallel straight lines Δ and Δ' with same orientation and such that $T \subset \Delta^-$ and $T_i \subset \Delta'^-$,
- and of edges of $\mathcal{C}^k_{S,v}$ that connect such vertices.

Using basic properties of convex hulls, it is easy to see that $\varphi(T_i)$ is a connected polygonal line and that:

Proposition 3 *If $(g(T_1), ..., g(T_m))$ is the counterclockwise-ordered sequence of vertices of $\mathcal{D}^k_{S,v}$, then $\mathcal{C}^k_{S,v} = (\varphi(T_1), ..., \varphi(T_m))$.*

3 Triangulating $\mathbf{g^k(V)}$

We show now that the centroids of the k-sets of a convex inclusion chain of V are the vertices of a triangulation of $g^k(V)$ that has some common characteristics with the order-k Delaunay diagram of V. Recall that this diagram is dual to the order-k Voronoi diagram and that its vertices are the centroids of the k-element subsets of V that determine the order-k Voronoi regions of V. The order-k Delaunay diagram is then a triangulation of $g^k(V)$ whose every edge $g(T)g(T')$ is such that $|T \cap T'| = k-1$ [7, 11]. We show now that the centroids of the k-sets of any convex inclusion chain of V are also the vertices of such a triangulation. From Proposition 1, we already know that the edges of every k-set polygon fulfill the property. Moreover:

Proposition 4 *For every vertex $g(T_i)$ of $\mathcal{D}^k_{S,v}$ and for every vertex $g(T)$ of $\varphi(T_i)$, there exists $s \in T_i$ such that $T = (T_i \setminus \{s\}) \cup \{v\}$ and the segments $g(T_i)g(T)$ triangulate the polygon $\mathcal{P} = \overline{g^k(S \cup \{v\}) \setminus g^k(S)}$.*

Proof. (i) By definition, for every vertex $g(T)$ of $\varphi(T_i)$, there exist two parallel oriented straight lines Δ and Δ' such that $\Delta^- \cap S = T_i$ and $\Delta'^- \cap (S \cup \{v\}) = T$. Thus, there is a unique point s of S between Δ and Δ' and we have $T = (T_i \setminus \{s\}) \cup \{v\}$.

(ii) Now, it is not difficult to show that $g(T)$ can be separated from $g^k(S)$ by a straight line and thus that $g(T_i)g(T) \subset \mathcal{P}$. Moreover, from Proposition 3, two such segments can only intersect at their endpoints.

(iii) The boundary of $\mathcal{P}$ is composed of the edges of $\mathcal{D}^k_{S,v}$, of the edges of $\mathcal{C}^k_{S,v}$, and of the two other edges to create. From Proposition 3, for every edge $g(T)g(T')$ of $\mathcal{C}^k_{S,v}$, there exists a unique $i \in \{1, ..., m\}$ such that $g(T)g(T')$ is an edge of $\varphi(T_i)$. The triangle $g(T)g(T')g(T_i)$ splits then $\mathcal{P}$ into two simple polygons. In the same way, if $g(T_i)g(T_{i+1})$ is an edge of $\mathcal{D}^k_{S,v}$, $\varphi(T_i)$ and $\varphi(T_{i+1})$ have a common vertex $g(T)$ and the triangle $g(T)g(T_i)g(T_{i+1})$ splits $\mathcal{P}$ into two simple polygons. By induction, $\mathcal{P}$ can thus be triangulated by such triangles (see Figure 3). □

Now, if $\mathcal{V} = (v_1, v_2, ..., v_n)$ is a convex inclusion chain of V, by applying Proposition 4 successively to the subsets $S_k = \{v_1, ..., v_k\}, ..., S_n = \{v_1, ..., v_n\}$, we get:

Theorem 5 *The centroids of the k-sets of $\mathcal{V}$ are the vertices of a triangulation of $g^k(V)$ whose every edge $g(T)g(T')$ is such that $|T \cap T'| = k-1$.*

The triangulation determined by this theorem is called the order-k triangulation of $\mathcal{V}$ and is denoted by $\mathcal{T}^k(\mathcal{V})$ (see Figure 4).

It is easy to see that, if the edges of a triangulation fulfill Theorem 5 then, for every triangle $g(T)g(T')g(T'')$ of this triangulation, either $|T \cap T' \cap$

$T''| = k-1$ (called a type-1 triangle), or $|T \cup T' \cup T''| = k+1$ (called a type-2 triangle).

4 Order-k triangulations and centroid triangulations

Lee has proposed an algorithm to construct the order-k Voronoi diagram by starting with the (order-1) Voronoi diagram and iteratively computing the order-i diagram from the order-$(i-1)$ diagram [7]. This algorithm can be dualized to construct iteratively the order-k Delaunay diagram starting with the (order-1) Delaunay diagram [11]. The method to construct the order-i Delaunay diagram from the order-$(i-1)$ diagram is the following:

Algorithm 1

- *For every type-1 triangle $g(P \cup \{r\})g(P \cup \{s\}) g(P \cup \{t\})$ of $Del_{i-1}(V)$ compute the triangle $g(P \cup \{r,s\})g(P \cup \{r,t\})g(P \cup \{s,t\})$.*
- *The set τ of these triangles is the set of type-2 triangles of $Del_i(V)$.*
- *The type-1 triangles of $Del_i(V)$ are obtained by computing the constrained (order-1) Delaunay triangulation of $\overline{g^i(V) \setminus \tau}$.*

In [8], Lyu and Snoeyink conjectured that, starting with any triangulation of the point set V and computing any constrained triangulation at every step, this algorithm constructs triangulations whose edges verify Theorem 5 (they proved the result for $k \leq 3$). The triangulations generated in this way are called centroid triangulations. Here we show that, for all k:

Theorem 6 *The order-k triangulation of any convex inclusion chain is a centroid triangulation.*

Proof. For every point set S we call (centroid) triangulation sequence of S, any sequence $(\mathcal{A}^1, ..., \mathcal{A}^{|S|})$ of centroid triangulations such that $\mathcal{A}^1$ is a triangulation of S and, for all $i \in \{2, ..., |S|\}$, $\mathcal{A}^i$ is obtained from $\mathcal{A}^{i-1}$ by the generalization of Algorithm 1. Note that $\mathcal{A}^1$ contains only type-1 triangles, $\mathcal{A}^{|S|-1}$ only type-2 triangles, and $\mathcal{A}^{|S|} = g^{|S|}(S)$ is reduced to the unique point $g(S)$.

Suppose by induction that, for every set S of $n-1$ points, for every convex inclusion chain $\mathcal{S}$ of S, and for every positive integer $k \leq n-1$, the order-k triangulation $\mathcal{T}^k(\mathcal{S})$ of $\mathcal{S}$ is a centroid triangulation of S and that $(\mathcal{T}^1(\mathcal{S}), ..., \mathcal{T}^{n-1}(\mathcal{S}))$ is a triangulation sequence of S. This is trivially true for $n-1=1$.

Let now v be a point not belonging to $conv(S)$, $V = S \cup \{v\}$, and $\mathcal{V} = (\mathcal{S}, v)$. When v is added to S, $\mathcal{C}^1_{S,v}$ is composed of the unique vertex v and $\mathcal{D}^1_{S,v}$ is composed of the edges and vertices of the boundary

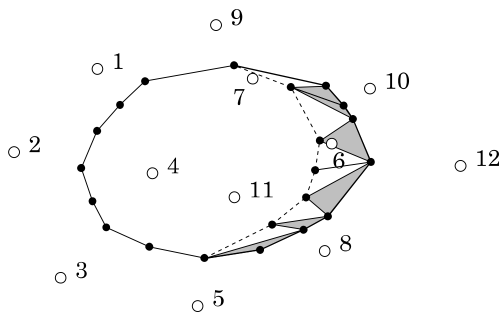

Figure 3: A triangulation of $\overline{g^4(S \cup \{12\}) \setminus g^4(S)}$, with $S = \{1, ..., 11\}$. The white triangles are of type 1 and the grey triangles are of type 2.

of $\mathcal{T}^1(\mathcal{S})$ visible from v. By connecting these vertices to v, we get $\mathcal{T}^1(\mathcal{V})$ which is the first element of a triangulation sequence of V. Assume now, as a second induction hypothesis, that for a positive integer $h \leq n-1$, $(\mathcal{T}^1(\mathcal{V}), ..., \mathcal{T}^{h-1}(\mathcal{V}))$ is an initial part of a triangulation sequence of V. $\mathcal{T}^h(\mathcal{V})$ verifies Theorem 5 and, from the proof of Proposition 4, $\overline{\mathcal{T}^h(\mathcal{V}) \setminus \mathcal{T}^h(\mathcal{S})}$ has two kinds of triangles: Triangles with one edge on $\mathcal{D}^h_{S,v}$ and the opposite vertex on $\mathcal{C}^h_{S,v}$, and triangles with one edge on $\mathcal{C}^h_{S,v}$ and the opposite vertex on $\mathcal{D}^h_{S,v}$ (see Figure 3). Using Proposition 4, it can be shown that these triangles are respectively of type 1 and 2. With the argument that $e_P(s,t)$ is an edge of $\mathcal{C}^h_{S,v}$ if and only if $e_{P \setminus \{v\}}(s,t)$ is an edge of $\mathcal{D}^{h-1}_{S,v}$, the type-2 triangles of $\overline{\mathcal{T}^h(\mathcal{V}) \setminus \mathcal{T}^h(\mathcal{S})}$ can be obtained from the type-1 triangles of $\overline{\mathcal{T}^{h-1}(\mathcal{V}) \setminus \mathcal{T}^{h-1}(\mathcal{S})}$ by the generalization of Algorithm 1. From the first induction hypothesis, $\mathcal{T}^h(\mathcal{V})$ can then also be obtained from $\mathcal{T}^{h-1}(\mathcal{V})$ by this algorithm and, from the second induction hypothesis, $(\mathcal{T}^1(\mathcal{V}), ..., \mathcal{T}^h(\mathcal{V}))$ is an initial part of a triangulation sequence of V, for all $h \in \{1, ..., n-1\}$. Since $\mathcal{T}^n(\mathcal{V})$ is reduced to the centroid $g^n(V)$, it follows that $(\mathcal{T}^1(\mathcal{V}), ..., \mathcal{T}^n(\mathcal{V}))$ is a triangulation sequence of V. □

5 Construction of a centroid triangulation

As we know from [7] and [5], the order-k Delaunay diagram and the order-k triangulations of convex inclusion chains have both $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} \gamma^j(V)$ vertices (with $\gamma^j(V)$ the number of j-sets of V and $\sum_1^0 = 0$). Lyu and Snoeyink have conjectured that any centroid triangulation has $O(k(n-k))$ vertices. Thus, the generalization of Algorithm 1 constructs a centroid triangulation in at least $O(n \log n + k^2(n-k))$ time (at least $O(n \log n)$ for the order-1 triangulation and at least $O(k(n-k))$ for each of the $k-1$ other centroid triangulations). We show now that a particular centroid triangulation can be constructed in $O(n \log n + k(n-k) \log^2 k)$ time.

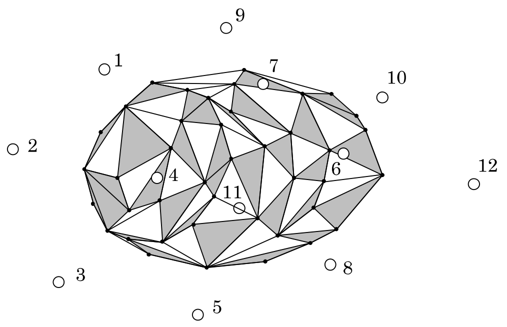

Figure 4: The order-4 triangulation of the convex inclusion chain $(2, 3, 1, 4, 5, 9, 11, 7, 8, 6, 10, 12)$.

To handle a centroid triangulation, we need to know its combinatorial structure, to maintain for every edge $g(P \cup \{s\})g(P \cup \{t\})$ a link to s and t, and to store the set T for exactly one vertex $g(T)$. Moreover, denoting by $|\mathcal{D}^k_{S,v}|$ and $|\mathcal{C}^k_{S,v}|$ the numbers of vertices of $\mathcal{D}^k_{S,v}$ and of $\mathcal{C}^k_{S,v}$, it has been shown in [5] that, if $g^k(S)$ and v are given, $\mathcal{D}^k_{S,v}$ and $\mathcal{C}^k_{S,v}$ can be found in $O(|\mathcal{D}^k_{S,v}| \log^2 k + |\mathcal{C}^k_{S,v}|)$ time, provided that one vertex $g(T_i)$ of $\mathcal{D}^k_{S,v}$ is known and that the convex hull of T_i is stored in a fully dynamic convex hull data structure (see [9]). Then we have:

Theorem 7 *V admits a centroid triangulation that can be constructed in $O(n \log n + k(n-k) \log^2 k)$ time.*

Proof. Let $\mathcal{V} = (v_1, ..., v_n)$ be a sequence of the points of V sorted by increasing x-coordinates. Obviously, $\mathcal{V}$ is a convex inclusion chain of V. For every subset $S_j = \{v_1, ..., v_j\}$ $(j \in \{k, ..., n-1\})$, it can be shown that $\mathcal{D}^k_{S_j, v_{j+1}}$ contains a vertex of $g^k(S_j)$ with maximal x-coordinate, i.e., the centroid of k points of S_j with maximal x-coordinates. If we maintain the dynamic convex hull of these k points and use the triangulation method of the proof of Proposition 4, given $g^k(S_j)$, a triangulation of $\overline{g^k(S_{j+1}) \setminus g^k(S_j)}$ can be constructed in $O(|\mathcal{D}^k_{S_j, v_{j+1}}| \log^2 k + |\mathcal{C}^k_{S_j, v_{j+1}}|)$ time.

Starting with $g^k(S_k) = g(S_k)$ and applying this triangulation method for all $j \in \{k, ..., n-1\}$, we get an order-k triangulation of $\mathcal{V}$ in $O(\sum_{j=k}^{n-1} |\mathcal{D}^k_{S_j, v_{j+1}}| \log^2 k + \sum_{j=k}^{n-1} |\mathcal{C}^k_{S_j, v_{j+1}}|)$ time. Now, $\sum_{j=k}^{n-1} |\mathcal{C}^k_{S_j, v_{j+1}}| + 1$ is the total number of vertices of the order-k triangulation of $\mathcal{V}$ and it is easy to see that $\sum_{j=k}^{n-1} |\mathcal{D}^k_{S_j, v_{j+1}}| < \sum_{j=k}^{n-1} |\mathcal{C}^k_{S_j, v_{j+1}}|$. Since the total number of vertices of the order-k triangulation of $\mathcal{V}$ is $O(k(n-k))$, this triangulation can be constructed in $O(k(n-k) \log^2 k)$ time, after having sorted V. $\square$

6 Conclusion

In this paper, we have shown that the family of centroid triangulations of a planar point set, which is known to contain the order k-Delaunay diagram, also contains the family of order-k triangulations of the convex inclusion chains of the point set.

Now, if we were able to show that all the centroid triangulations have the same number of vertices, this would completely explain why the number of k-sets of a convex inclusion chain is equal to the number of regions of the order-k Voronoi diagram. To achieve this goal, we will probably need to find a geometric characterization of centroid triangulations.

References

[1] A. Andrzejak and K. Fukuda. Optimization over k-set polytopes and efficient k-set enumeration. In *Proc. 6th Workshop Algorithms Data Struct.*, volume 1663 of *Lecture Notes Comput. Sci.*, pages 1–12. Springer-Verlag, 1999.

[2] A. Andrzejak and E. Welzl. In between k-sets, j-facets, and i-faces: (i, j)-partitions. *Discrete Comput. Geom.*, 29:105–131, 2003.

[3] F. Aurenhammer and O. Schwarzkopf. A simple online randomized incremental algorithm for computing higher order Voronoi diagrams. *Internat. J. Comput. Geom. Appl.*, 2:363–381, 1992.

[4] T. K. Dey. Improved bounds on planar k-sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.

[5] W. El Oraiby and D. Schmitt. k-sets of convex inclusion chains of planar point sets. In *Proc. 31st Intern. Symp. Math. Found. Comput. Sci.*, volume 4162 of *LNCS*, pages 339–350. Springer-Verlag, 2006.

[6] J. Gudmundsson, M. Hammar, and M. J. van Kreveld. Higher order Delaunay triangulations. *Comput. Geom.*, 23(1):85–98, 2002.

[7] D. T. Lee. On k-nearest neighbor Voronoi diagrams in the plane. *IEEE Trans. Comput.*, C-31:478–487, 1982.

[8] Y. Liu and J. Snoeyink. Quadratic and cubic b-splines by generalizing higher-order Voronoi diagrams. In *Symposium on Computational Geometry*, pages 150–157, 2007.

[9] M. H. Overmars and J. van Leeuwen. Maintenance of configurations in the plane. *J. Comput. Syst. Sci.*, 23:166–204, 1981.

[10] G. W. Peck. On k-sets in the plane. *Discrete Math.*, 56:73–74, 1985.

[11] D. Schmitt and J.-C. Spehner. On Delaunay and Voronoi diagrams of order k in the plane. In *Proc. 3rd Canad. Conf. Comput. Geom.*, pages 29–32, 1991.

[12] G. Tóth. Point sets with many k-sets. *Discrete Comput. Geom.*, 26(2):187–194, 2001.