Triangulations of Line Segment Sets in the Plane

Mathieu Brévilliers, Nicolas Chevallier, and Dominique Schmitt

Laboratoire MIA, Université de Haute-Alsace
4, rue des Frères Lumière, 68093 Mulhouse Cedex, France
{Mathieu.Brevilliers,Nicolas.Chevallier,Dominique.Schmitt}@uha.fr

Abstract. Given a set S of line segments in the plane, we introduce a new family of partitions of the convex hull of S called segment triangulations of S. The set of faces of such a triangulation is a maximal set of disjoint triangles that cut S at, and only at, their vertices. Surprisingly, several properties of point set triangulations extend to segment triangulations. Thus, the number of their faces is an invariant of S. In the same way, if S is in general position, there exists a unique segment triangulation of S whose faces are inscribable in circles whose interiors do not intersect S. This triangulation, called segment Delaunay triangulation, is dual to the segment Voronoi diagram. The main result of this paper is that the local optimality which characterizes point set Delaunay triangulations [10] extends to segment Delaunay triangulations. A similar result holds for segment triangulations with same topology as the Delaunay one.

1 Introduction

The Voronoi diagram of a set S of sites in the d-dimensional Euclidean space $\mathcal{E}$ partitions $\mathcal{E}$ into regions, one per site; the region for a site s consists of all points closer to s than to any other site. In very recent years, particular attention has been paid to the study of the Voronoi diagram of a set of line segments in three dimensions [13], [18], [9], ... However, the topology of this diagram is really known only for a set of three lines [8]. The investigation for the point set Voronoi diagram has been fairly facilitated by the well understanding of its dual, the Delaunay diagram. Recall that, if no $d+1$ points of S are cospherical, the Delaunay diagram of S is the unique triangulation of S whose tetrahedra are inscribable in empty spheres, that is, spheres whose interiors do not intersect S. Among all the triangulations of S, the Delaunay diagram of S has many optimality properties, some of them extending in any dimension [15], [17]. Until now, no such properties have been given, even in the plane, for the dual of the segment Voronoi diagram which has been introduced by Chew and Kedem [5]. Surprisingly, no family of diagrams containing this dual diagram has been defined whereas many generalizations of point set triangulations have been studied: constrained triangulations [11], pseudo-triangulations [16], pre-triangulations [1], ...

In this paper, we introduce a new family of diagrams, called segment triangulations, which decompose the convex hull of a set S of points and line segments

V. Arvind and S. Prasad (Eds.): FSTTCS 2007, LNCS 4855, pp. 388–399, 2007.

in the plane. The set of faces of a segment triangulation of S is a maximal set of disjoint triangles such that the vertices of each triangle belong to three distinct sites of S and no other point of the triangle belongs to S. The edges of the segment triangulation are the (possibly two-dimensional) connected components of the convex hull of S when the sites and open faces are removed. These definitions are natural for, when S is a point set we recover the definitions of the faces and the edges of a point set triangulation. The aim of this paper is to study this new kind of triangulation in order to characterize by local properties the dual of the segment Voronoi diagram among the set of segment triangulations.

The segment triangulations are studied for their own sake in the two first sections. We show that they retain different geometrical and topological properties of point set triangulations and that they are intimately related to some generalized constrained triangulations.

In the next section, we prove that there exists one and only one segment triangulation of S whose faces are inscribable in empty circles. We show that this triangulation, called segment Delaunay triangulation, is the dual, introduced by Chew and Kedem, of the segment Voronoi diagram and can thus be constructed in $O(n \log n)$ time.

The point set Delaunay triangulation admits an important local characterization which is used to prove many of its optimality properties and enables to check in linear time whether a given triangulation is Delaunay or not. This local property states that a triangulation is Delaunay if and only if every couple of faces sharing a common edge is in Delaunay position with respect to its four defining sites [10]. The main result of the second part of the paper is that this property also characterizes the segment Delaunay triangulation among all the segment triangulations of a given set of line segments. We also give another local property that characterizes the set of segment triangulations having the same topological structure as the segment Delaunay triangulation.

2 Segment Triangulations and Constrained Triangulations

Let S be a finite set of $n \geq 2$ disjoint closed segments in the plane, which we call sites. Throughout this paper, a closed segment may possibly be reduced to a single point. We say that a circle is tangent to a site s if s meets the circle but not its interior. The sites of S are supposed to be in general position, that is, we suppose that no three segment endpoints are collinear and that no circle is tangent to four sites.

Definition 1. *A segment triangulation P of S is a partition of the convex hull $conv(S)$ of S in disjoint sites, edges and faces such that:*

(i) Every face of P is an open triangle whose vertices belong to three distinct sites of S and whose open edges do not intersect S,
(ii) No face can be added without intersecting another one,
(iii) The edges of P are the (possibly two-dimensional) connected components of $conv(S) \setminus (F \cup S)$, where F is the set of faces of P.

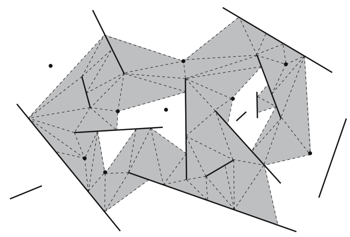

Fig. 1. A weakly constrained triangulation (dotted lines) of an S-polygon (in grey)

Such a triangulation always exists, that is, for any set S, there is a finite number of faces verifying Definition 1. Indeed, it is not difficult to see that at most two disjoint triangles can have their vertices on the same three sites (see Figure 3(a)).

There is a well-known triangulation defined on a set of points and line segments: The constrained triangulation. It is a triangulation of the set of points and segment endpoints such that every given line segment is a side of a triangle. These triangulations are mainly used to triangulate terrains with topographic constraints (mountain crests, roads, ...) or interiors of polygons. However, the triangles being mostly too irregular, so called Steiner points are added to the initial point set (see for example [3]). Steiner points added on the segments enable to split them into subsegments and to generate a better constrained triangulation. We show now that segment triangulations are intimately related to a kind of generalized Steiner triangulation that we call weakly constrained triangulation. In this triangulation (see Figure 1), a point added on a line segment does not split the segment but becomes a vertex of triangles that are on one side of the segment. Therefore, the two sides of a segment are independent. This enables, for example, to independently triangulate two slopes on both sides of a same mountain crest. We now define the weakly constrained triangulation of a restricted region.

Definition 2

(i) Given a set S of sites, we call S-polygon (possibly with holes), any closed two-dimensional subset A of $conv(S)$, equal to the closure of its interior, such that $A \setminus S$ is connected and the boundary of A is composed of a finite number of disjoint line segments that are of the two following forms:

- *closed and contained in S,*
- *open, not intersecting S, and with their endpoints in S.*

(ii) We call weakly constrained triangulation of A (with respect to S), any partition of A in triangles whose vertices belong to S, whose interiors do not cut S, and whose open sides either do not cut S or are contained in S.

When $A = conv(S)$, such a triangulation is also called a weakly constrained triangulation of S.

The following lemma will enable to establish the connection between segment triangulations and weakly constrained triangulations.

Lemma 1. *If A is an S-polygon that intersects at least three sites of S then every weakly constrained triangulation of A contains at least one triangle whose vertices belong to three distinct sites of S.*

Proof. Given a weakly constrained triangulation T of A, let $\Delta_T(A)$ be the (possibly empty) set of triangles of T having one side in S. We show, by induction on the number $|\Delta_T(A)|$ of triangles of $\Delta_T(A)$, that T contains at least one triangle whose vertices belong to three distinct sites of S.

Obviously, if $\Delta_T(A) = \emptyset$, the vertices of every triangle of T belong to three distinct sites. Suppose now the result true for every weakly constrained triangulation T with $|\Delta_T(A)| < k$ $(k \geq 1)$.

For every weakly constrained triangulation T of A with $|\Delta_T(A)| = k$ and for every closed triangle t of $\Delta_T(A)$, the closure $\overline{A \setminus t}$ of $A \setminus t$ intersects the same sites as A. If $A \setminus t$ is connected, $A' = \overline{A \setminus t}$ is an S-polygon. Otherwise, $A \setminus t$ has two connected components, the closure of at least one of them being an S-polygon. In the latter case, each of the S-polygons intersects the two sites to which the vertices of t belong. It follows that at least one of these S-polygons intersects at least three sites. Let A' be this S-polygon. In both cases, if T' is the restriction of T to A', $|\Delta_{T'}(A')| < |\Delta_T(A)|$. Thus, by induction hypothesis, T' contains at least one triangle whose vertices belong to three distinct sites of S. It is the same for T. □

It follows from this lemma that, in any weakly constrained triangulation of S, the set F of triangles having their vertices on three distinct sites of S is maximal. Indeed, the closure of every connected component e of $conv(S) \setminus (F \cup S)$ is either a line segment connecting two points of S or an S-polygon. In the second case, it follows from Lemma 1 that $\overline{e}$ can only intersect two sites. Therefore no triangle having its vertices on three distinct sites of S can be added without cutting $F \cup S$. Thus, the theorems:

Theorem 1. *Every weakly constrained triangulation of S is a refinement of a segment triangulation of S, that is, a segment triangulation whose edges are decomposed in triangles.*

Theorem 2. *The closure of every edge of a segment triangulation of S intersects exactly two sites of S.*

This shows that an edge of a segment triangulation P of S is really an edge in that sense that it "connects" exactly two sites of S. Its shape can also be deduced from the discussion above. The closure of an edge either is reduced to a line segment joining two points in two distinct sites of S, or is a triangle with one side and its opposite vertex in S, or is a (possibly non-convex) quadrilateral with two opposite sides in S (see Figure 2). Moreover, every edge of P contains

- either two sides of two triangles of P,
- or one side of one triangle of P and one side of $conv(S)$ that is not a site,
- or two such sides of $conv(S)$.

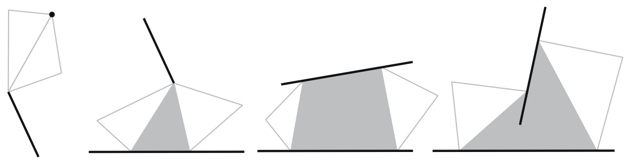

Fig. 2. Examples of edges (grey) connecting two sites in a segment triangulation

Here the edges of a segment triangulation are implicitly defined by complementarity with respect to the faces and to the sites. If we want to extend segment triangulations in d dimensions, the faces of dimension less than d have to be defined explicitly. In the plane, this could be done by defining the edges in the following way: Take a maximal set E of open disjoint line segments that do not cut S and whose endpoints belong to S (in general E is infinite). Then, it can be proved that the connected components of E are the edges of a segment triangulation of S.

3 Topological Properties of Segment Triangulations

Since every edge of a segment triangulation P of S "connects" two sites of S, we can associate an abstract graph with P such that:

- the vertices of the graph are the sites of S,
- the edges connecting two sites s and t in the graph are the edges of P whose closures intersect s and t.

Proposition 1. *The abstract graph associated with a segment triangulation P of S is planar.*

Proof. For every site s of S, let γ_s be a convex closed Jordan curve such that:

- s is inside γ_s (i.e. in the subset of the plane bounded by γ_s),
- $S \setminus s$ is outside γ_s,
- the interior of γ_s intersects only the edges of P whose closures intersect s.

Replace now every site s by a point p_s inside γ_s. For every edge e of P that intersects γ_s, replace the subset of e inside γ_s by a line segment connecting p_s to a point of e on γ_s. While doing this, the order of the edges around s remains unchanged and the reduced edges do not intersect. Once this transformation is fulfilled in every Jordan curve γ_s, replace every reduced edge by a Jordan arc included in it. Finally, we get a planar representation G of the abstract graph associated with P (see Figure 3(b)). $\square$

Theorem 3. *Every segment triangulation P of a set S of n sites contains $3n - n' - 3$ edges and $2n - n' - 2$ faces, where n' is the number of edges of $conv(S)$ that are not sites.*

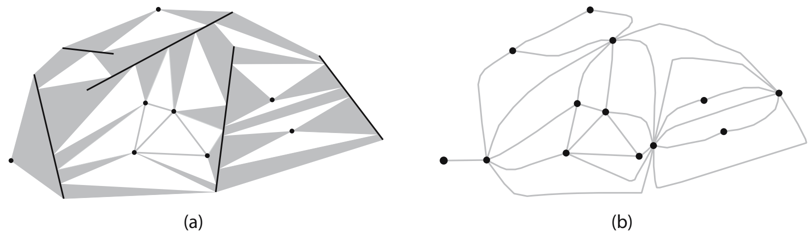

Fig. 3. A segment triangulation (a) (the sites are in black, the edges in grey, and the faces in white) and its associated graph (b)

Proof. Counting the edges and faces of P comes down to counting the edges and bounded faces of the planar representation G constructed in the proof of Proposition 1. Moreover, the unbounded face of G corresponds to the complementary of $conv(S)$. The result is then an immediate consequence of Euler's relation, of the fact that every bounded face of G has three edges, and that the edges adjacent to one (resp. no) bounded face appear once (resp. twice) while traversing the boundary of the unbounded face of G. □

An interesting consequence of this theorem is that the size of a segment triangulation is linear with the number of sites. Moreover, it shows that the number of triangles of the triangulation is an invariant of the set of sites. This is an extension of a well-known property of the triangulations of planar point sets.

Using the planar representation G constructed in the proof of Proposition 1, we can associate a combinatorial map M with the segment triangulation P:

– the underlying graph is the abstract graph associated with the triangulation P,
– for every vertex s of M, the cyclic ordering of the edges out of s agrees with the counter-clockwise ordering of the associated Jordan arcs around s in the planar representation G.

Note that, in general, the same map M is associated with different segment triangulations of S. We say that:

Definition 3. *Two segment triangulations of S have the same topology if they have the same associated combinatorial map.*

In order to use M as a data structure to store the segment triangulation P, we only need to add the coordinates of the vertices of the triangles of P in the structure: One vertex per oriented edge. A segment triangulation of a set S of n sites can thus be stored using $O(n)$ space. Furthermore, from Theorem 1, every constrained triangulation of S is a refinement of a segment triangulation of S. There exists a sweep-line algorithm to construct a constrained triangulation in $O(n \log n)$ time [7] and this algorithm can easily be adapted to construct a segment triangulation also in $O(n \log n)$ time.

4 Segment Delaunay Triangulation and Segment Voronoi Diagram

Among the set of all segment triangulations, some are distinguished. For example, we could look for the segment triangulation whose faces have a maximal total area. Here we will be interested in the segment triangulation whose faces are inscribable in empty circles. In this section, we prove the existence and unicity of this special segment triangulation and we show that it is dual to the segment Voronoi diagram (see Figure 4). Our proof uses some properties of the segment Voronoi diagram, which can be found in [2], [4], and [14].

Let now F be the set of triangles of the plane such that the vertices of each triangle belong to three distinct sites of S and such that the interior of the circumcircle of each triangle does not intersect S.

Theorem 4

(i) The triangles of F are the faces of a segment triangulation P of S, which we call the segment Delaunay triangulation.

(ii) The combinatorial map M associated with P is dual to the segment Voronoi diagram of S.

Proof. Since the interior of the circumcircle of every triangle of F is empty, two such triangles cannot intersect. Thus, they are faces of a segment triangulation. On the one hand, the number of vertices of the Voronoi diagram $Vor(S)$ of S is known and by Theorem 3, it is the same as the number of triangles of a segment triangulation of S. On the other hand, each vertex of the Voronoi diagram corresponds to one triangle of F. Therefore, the number of triangles of F is maximal, which means that F is the set of triangles of a segment triangulation P. Furthermore, by definition of the Voronoi diagram, there is a one-to-one correspondence between the regions of $Vor(S)$ and the sites, which are, by definition, the vertices of M.

It remains to study the edges of M and of $Vor(S)$. Let a be an edge of $Vor(S)$ incident to the two Voronoi regions of s and t. Each point p in a is the center of an empty circle C_p touching the two sites s and t at the points p_s and p_t. It is not difficult to prove that such an open segment p_sp_t never meets a triangle

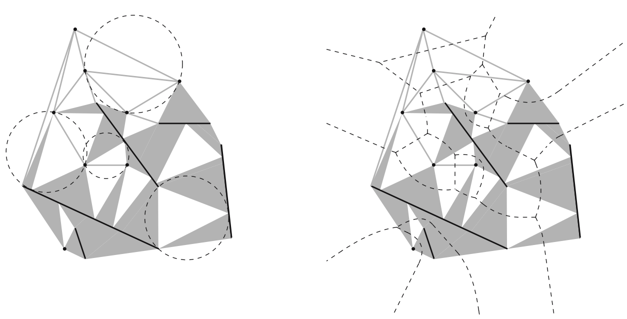

Fig. 4. A segment Delaunay triangulation and an illustration of the duality

of F. Thus, for each p in a, the open segment p_sp_t is included in an edge of the segment triangulation P. Furthermore, the union E_a of all the open segments p_sp_t, $p \in a$, is a connected subset of $conv(S)$, therefore E_a is included in a single edge e of P, which is incident to s and t. The last thing to see is that for each edge e of P there is exactly one edge a of $Vor(S)$ such that $E_a \subset e$. Since the numbers of edges of P and of $Vor(S)$ are equal, it suffices to prove that for each edge e of P there is at least one edge a such that $E_a \subset e$. Now, any boundary segment of an edge e linking two sites s and t, is of the previous kind p_sp_t. Therefore there is an edge a of $Vor(S)$ such that $E_a \subset e$. □

It is easy to see that the segment Delaunay triangulation of S defined in this theorem is equivalent to the dual of $Vor(S)$ introduced by Chew and Kedem, which they called the edge Delaunay triangulation of S [5]. Using algorithms that construct segment Voronoi diagrams, the segment Delaunay triangulation can be computed in $O(n \log n)$ time [14].

5 Legality in Segment Triangulations

An interesting property of the Delaunay triangulation of a planar point set is the legal edge property. Consider an edge of a point set triangulation and its two adjacent triangles. The edge is illegal if a vertex of one of these triangles lies inside the circumcircle of the other triangle. It is well-known that the Delaunay triangulation of a point set is the unique triangulation of this point set without illegal edge. In the following, we are going to prove a similar property for segment triangulations.

Definition 4. *An egde of a segment triangulation is legal if the circumcircles of its adjacent triangles contain no point of the sites adjacent to these triangles in their interiors.*

Theorem 5. *The segment Delaunay triangulation of S is the unique segment triangulation of S whose all edges are legal.*

Proof. Obviously, the segment Delaunay triangulation has no illegal edge. Let P be a segment triangulation which is not Delaunay and let f be a face of P whose circumcircle c_f contains a point of S in its interior d_f. We have to prove that P has an illegal edge. Let x be a point in f and p a point in d_f lying on a site. We can assume that the interior of the segment xp does not intersect S. Denote by k the number of edges crossed by the segment xp. Note that $k \geq 1$, for, by definition, p can neither be in f, nor in an edge adjacent to f. Denote e the first edge crossed by xp, g the other face adjacent to e, c_g its circumcircle, d_g the interior of c_g, ab the side of g contained in e, and u the site that contains the vertex of g that is not a vertex of e (see Figure 5). If $k = 1$, p lies on u and therefore the edge crossed by xp is illegal. Now suppose that, if xp crosses k edges then at least one of them is illegal. We have to prove that if xp crosses $(k+1)$ edges then P has an illegal edge. If the edge e is illegal we are done. Otherwise

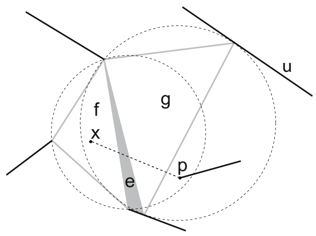

Fig. 5. Illustration of the proof of Theorem 5

the points a and b cannot be in the disk d_f. Moreover the point $y = ab \cap xp$ is in d_f. Therefore, the segment ab splits d_f into two parts. Denote d_1 the part containing the face f and d_2 the other part. The disk d_g must contain at least d_1 or d_2, and since e is legal it can not contain d_1. It follows that the segment yp is in d_g and crosses one edge less than xp. Using the induction hypothesis, we conclude that P has an illegal edge. $\square$

As remarked in section 3, different segment triangulations of S can have the same topology. Especially an infinite number of segment triangulations of S have the topology of the segment Delaunay triangulation of S. As the segment Delaunay triangulation can be easily computed when its topology is known, it is useless to store the coordinates of the vertices, which, moreover, are usually inexact. Thus it is interesting to know if a given segment triangulation of S has the topology of the segment Delaunay triangulation of S. Furthermore, suppose that a segment triangulation of S is Delaunay and that the sites of S are slightly moved. Then we can wonder if the initial topology remains the topology of the segment Delaunay triangulation of the new set S. For these reasons, we define the edge legality for maps associated with segment triangulations.

Definition 5. *Let f be a face of a segment triangulation of S. The tangency triangle of f is the triangle such that:*

- *its vertices are on the same three sites as the vertices of f,*
- *the interior of its circumcircle does not intersect these three sites,*
- *if f and its tangency triangle are traversed in counter-clockwise direction, they encounter these three sites in the same order.*

Definition 6. *Let M be a map associated with a segment triangulation of S. An edge e of M is legal in the two following cases:*

1. *e is adjacent to at most one internal triangle.*
2. *e is adjacent to two internal triangles T_1 and T_2 and the following property holds. Denote t, r, u, v the sites such that t, r, u are incident to T_1 and r, t, v are incident to T_2 in counter-clockwise direction. Let $t_1r_1u_1$ and $r_2t_2v_2$ be the tangency triangles of T_1 and T_2 with $t_i \in t$, $r_i \in r$, $u_1 \in u$, and $v_2 \in v$. Then:*

- *The polygon $t_1t_2r_2r_1$ is either reduced to a segment or is a counter-clockwise oriented simple polygon (with three or four edges),*
- *The circumcircles' interiors of $t_1r_1u_1$ and $r_2t_2v_2$ do not intersect the sites t, r, u, v.*

Theorem 6. *Let M be a map associated with a segment triangulation P of S. Suppose that all the edges of this map are legal, then M is also the map associated with the segment Delaunay triangulation of S.*

Proof (sketch). We want to prove that the collection of tangency triangles gives rise to the segment Delaunay triangulation. Making use of previous theorem, we see that the only thing to prove is that the interiors of the tangency triangles are the faces of a segment triangulation of S.

The main idea of the proof is to use a result of Devillers et al. [6] which asserts that a representation of a combinatorial map by smooth curves in the plane is a planar graph if:

- All the circuits of the map are represented by simple closed curves,
- The ordering at each vertex s of the map is given by the geometric ordering of the curves emanating from the point representing s.

Actually, the result of Devillers et al. is stated with segments instead of smooth curves but an approximation argument leads to the same result for smooth curves.

First, for each $\varepsilon > 0$ sufficiently small, it is possible to construct a planar graph as done in Figure 6(a). All edges of this graph are smooth curves that are at a distance less than ε either from the sites or from the sides of the triangles of P. This planar graph is a representation in the plane of a new combinatorial map M' which does not depend on ε.

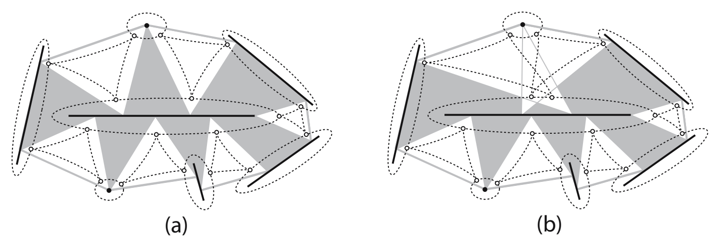

Fig. 6. (a) Planar graph deduced from P. (b) A new representation of the map M'.

Next, moving all the triangles T of P to their tangency positions T', we can define a new representation of the map M':

- The curves associated with each triangle of P moves from the initial triangle to the tangency triangle.

– The new closed curves around the sites are slightly more difficult to define. Suppose that T_1 and T_2 are two adjacent triangles of P incident to a site s. Call γ_s the "old" curve around s. There is a point p_i on γ_s associated with the vertex of T_i lying on s and there is a point p'_i on γ_s associated with the vertex of the tangency triangle T'_i lying on s. In the new representation of the map M', we take the portion of the curve γ_s going from p'_1 to p'_2 turning around s in the same direction as the portion of γ_s going from p_1 to p_2 (see Figure 6(b)).

This process ensures that the geometric ordering of the curves emanating from a vertex are the same for the old and the new representation of the map M'. Finally, thanks to the legality of all the edges, one can prove that the new representation of the circuits of M' are simple closed curves. Then, it follows by the result of Devillers et al. that the new representation of M' is a planar graph. Letting ε going to zero, we see that the tangency triangles are the faces of a segment triangulation. □

Theorem 6 enables to test whether a segment triangulation has the topology of the segment Delaunay triangulation by checking the edge legality. From Theorem 3, the number of edges is in $O(n)$ where $n = card(S)$, thus this test can be done in $O(n)$ time. Hence:

Corollary 1. *There is a linear time algorithm that checks whether a given segment triangulation has the same topology as the segment Delaunay triangulation.*

By duality this allows to check in linear time the correctness of the topology of a segment Voronoi diagram computed by a program. For more details on efficient program checkers in computational geometry see, for example, [6] and [12].

6 Conclusion

In this paper, we have notably shown that the segment Delaunay triangulation is the unique segment triangulation that is locally Delaunay in all its edges. As for point set triangulations, this should enable to prove optimality properties of the segment Delaunay triangulation and to give a flip algorithm that transforms any segment triangulation in the segment Delaunay triangulation by a sequence of local improvements. Together with this local characterization, there is a strong hint which makes us believe that a kind of flip algorithm should work with segment triangulations. Lifting a set of sites S onto the paraboloid $z = x^2 + y^2$, it is not hard to see that the triangles of the segment Delaunay triangulation are exactly the downward projection of the triangular faces of the lower convex hull of the lift of S; whereas the lift of any non-Delaunay face is above this lower convex hull, as in the case of point set triangulations. At last, we mention two possible extensions of segment triangulations. On the one hand, it is possible to define triangulations for a set S of disjoint compact convex subsets in the plane. We think that most of the results of this paper might extend to this more general setting. On the other hand, we hope that segment triangulations can be defined in higher dimensions and that it will help to better understand the topological structure of the segment Voronoi diagram in higher dimensions.

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