Two-dimensional line space Voronoi Diagram

Stéphane Rivière and Dominique Schmitt
Équipe Mage–LMIA– Université de Haute-Alsace
4 rue des Frères Lumière
68093 Mulhouse Cedex – France
stephane.riviere@uha.fr, dominique.schmitt@uha.fr

Abstract

Given a set of points called sites, the Voronoi diagram is a partition of the plane into sets of points having the same closest site. Several generalizations of the Voronoi diagram have been studied, mainly Voronoi diagrams for different distances (other than the Euclidean one), and Voronoi diagrams for sites that are not necessarily points (line segments for example).

In this paper we present a new generalization of the Voronoi diagram in the plane, in which we shift our interest from points to lines, that is, we compute the partition of the set of lines in the plane into sets of lines having the same closest site (where sites are points in the plane). We first define formally this diagram and give first properties. Then we use a duality relationship between points and lines to visualize this data structure and give more properties. We show that the size of this line space Voronoi diagram for n sites is in $\Theta(n^2)$ and give an optimal algorithm for its explicit computation.

We then show a remarkable relationship between this diagram and the dual arrangement of the sites and give a new result on an arrangement of lines: we show that the size of the zone of a line augmented with its incident faces is still in $O(n)$. We finally apply this result to show that the size of the zone of a line in the line space Voronoi diagram is in $O(n)$.

1. Introduction

Given a set of points called sites, the well-known Voronoi diagram represents the sets of points closer to a site than to the other ones. Several generalizations of this diagram have already been studied (see e.g. [2] and [7] for more informations). A first category of generalizations consists in studying different ways of computing the distance between the points and the sites (other metric than the classical euclidean one, weighted distances, ...). Another category of generalizations consists in extending the types of the sites (Voronoi diagram for line segments, for circles, ...). We propose here a new generalization of the Voronoi diagram in the plane by shifting our interest from points to lines, that is, we compute the sets of lines closer to a site than to the other ones.

We give a formal definition of this diagram and study its properties.

Like many problems which deals with sets of lines, we use a duality relationship between lines and points (see [3] and [5]) to visualize more easily the components of the line space Voronoi diagram. We can then give more properties of the line space Voronoi diagram, compute its size, and give an optimal algorithm for its computation.

Finally we show that the line space Voronoi diagram has an interesting relationship with the dual arrangement of the sites, that is, the arrangement of the dual lines of the sites. The arrangement of lines in the plane is the decomposition of the plane induced by the lines into faces, edges, and vertices and has already received a lot of attention (see [1] for a survey). In particular, the zone theorem states that the sum of the sizes of the faces of an arrangement cut by a line is linear. We extend this theorem by including the faces incident to the ones cut by the line, and show that the overall size remains linear. Finally we prove a zone theorem for the line space Voronoi diagram.

2. Definition and first properties of the line space Voronoi diagram

Given n points in the plane, called sites, we want to compute for each site the set of lines closer to the site than to the other ones. For the sake of simplicity we will assume that sites are in general position, that is, that no more than two sites are on a same line and that among the lines passing through two sites, none are parallel: complexity is maximal when sites are in general position.

We first state the *empty strip property* which will be use-

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ful in several proofs (in this paper, a strip will always denote an open strip and "a line is closer to a site" will mean that the line is closer to the site than to the other sites):

Property 1 *Given a set of sites, a line l is strictly closer to a site p if and only if the strip delimited by the parallel to l passing through p and by its symmetric according to l is empty of sites (figure 1).*

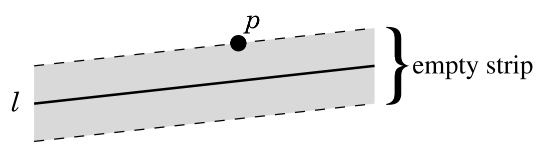

Figure 1. Empty strip property when l is closer to p

We can now define the *two-dimensional line space Voronoi Diagram*:

Definition 1 *Given n sites in the plane, we define the two-dimensional line space Voronoi Diagram as being the partition of the set of lines in the plane into faces: maximal connected components of lines strictly closer to a site, edges: maximal connected components of lines equidistant to two sites and strictly closer to these two sites (without passing through them), and vertices: lines passing through two sites and lines equidistant to three sites and strictly closer to these three sites.*

Let us take a closer look at this definition and this diagram (which will be called *line space Voronoi diagram* for short in the remainder). A line strictly closer to a site p can be moved with two degrees of liberty (for example translation and rotation) while remaining strictly closer to this site p, that is why we call the sets of such lines *faces*. We will say that a face made of lines strictly closer to a site p is *associated to* the site p. We can give the following characterization of a face:

Property 2 *Two lines belong to a same face of the line space Voronoi diagram if and only if they are strictly closer to a same site and separate the other sites into the two same sets.*

Proof: If the sets of sites above and below the two lines are different, then we cannot move continuously one line to the other without passing through a site and so become closer to this site. □

Given two sites, the intersection of the two sets of lines (non strictly) closer to each site is the set of lines equidistant to the two sites. There are two sorts of lines equidistant to two points:

Property 3 *A line is equidistant to two points p and q if and only if it either is parallel to the line (pq), or passes through the midpoint m_{pq} of p and q.*

From this property and the empty strip property we can characterize the edges of the line space Voronoi diagram:

Property 4 *Let p and q be two sites and m_{pq} their midpoint.*

A line l parallel to the line (pq) belongs to an edge of the line space Voronoi diagram if and only if the strip delimited by (pq) and by its symmetric according to l is empty of sites (figure 2a).

A line l passing through m_{pq} belongs to an edge of the line space Voronoi diagram if and only if the strip delimited by the lines parallel to l and passing respectively through p and q is empty of sites (figure 2b).

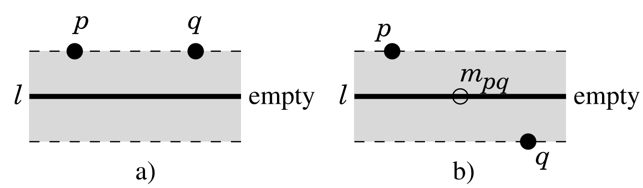

Figure 2. Line l belonging to an edge: a) parallel to two sites b) passing through the midpoint of two sites

We can move these lines with only one degree of liberty (translation in the first case, rotation around the midpoint in the second case), that is why we have called the sets of such lines *edges*.

By studying how a line can pass through a midpoint and either pass through another midpoint or be parallel to two sites, we can see that the intersection of two sets of lines equidistant to two sites is either a line equidistant to three sites p, q, r, or a line passing through two sites p and q.

From the properties of the edges and the empty strip property we can characterize the vertices of the line space Voronoi diagram:

Property 5 *The vertices of the line space Voronoi diagram are the lines (pq) passing through two sites (figure 3a) and the lines $(m_{pr}m_{qr})$ passing through two midpoints such that the strip delimited by (pq) and by the line parallel to (pq) passing through r is empty of sites (figure 3b).*

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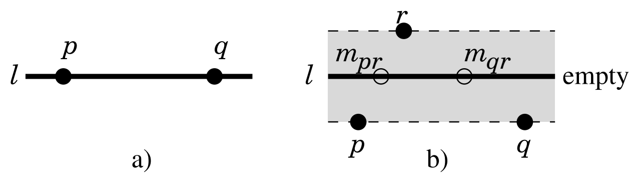

Figure 3. Line l being a vertex: a) passing through two sites b) equidistant to three sites

The first case, a line (pq), is the intersection of two sets of lines: the set of lines parallel to (pq) and the set of lines passing through the midpoint m_{pq}. We will see that this kind of vertex is incident to four edges.

The second case, a line $l = (m_{pr} m_{qr})$ passing through two midpoints of three sites, is the intersection of three sets of lines: the set of lines passing through m_{pr}, the set of lines passing through m_{qr}, and the set of lines parallel to (pq). We will see that this kind of vertex is incident to three edges.

3. The line space Voronoi diagram in dual space

3.1. Duality and line space Voronoi diagram for two sites

Sets of lines are difficult to represent and manipulate directly in the plane, so we will use a duality relationship between points and lines and work in the dual space where things will be clearer and proofs easier to give (see e.g. [3] for some uses of duality).

We use the following classical duality relationship between points and lines (which we orient from left to right): a line $l : y = ax + b$ is transformed into the dual point $l^*(a, -b)$, and a point $p\,(a, b)$ is transformed into the dual line $p^* : y = ax - b$ (vertical lines correspond to points at $-\infty$).

This duality relationship is an involution which preserves incidence relationships between points and lines, that is, the point p is above (resp. below, resp. on) the line l if and only if the dual point l^* is above (resp. below, resp. on) the dual line p^*.

In the dual space, lines become points, and sets of points are easier to deal with than sets of lines. In particular, the set of lines l_p passing through a given point p becomes a set of dual points l_p^* which is exactly the dual line p^*. We can also notice that a set of parallel lines becomes in the dual space a set of points on a vertical line. Finally, the dual line m_{pq}^* passes through the intersection of p^* and q^* and is vertically halfway between p^* and q^* (i.e., for any vertical line l_v, $m_{pq}^* \cap l_v$ is the midpoint of $p^* \cap l_v$ and $q^* \cap l_v$).

In the simplest case of two sites p and q, the line space Voronoi diagram in the dual space is made of four faces delimited by the dual line m_{pq}^* of the midpoint m_{pq} of $[pq]$ and the vertical line $//_{pq}^*$ corresponding to the lines parallel to (pq) (figure 4).

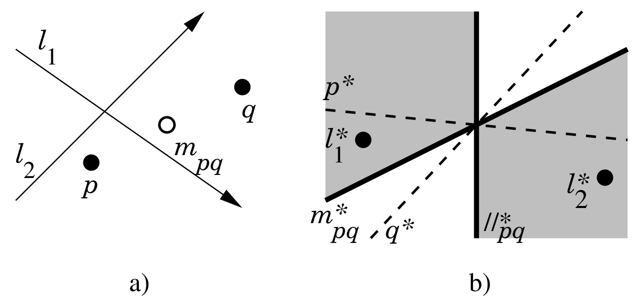

Figure 4. a) Lines closer to p b) line space Voronoi diagram of p and q in the dual space and dual points of the lines

In figure 4, the two gray (resp. white) faces correspond to lines closer to p (resp. q) than to the other site. The upper left gray face represents lines having m_{pq} above them and a slope smaller than the slope of (pq) (e.g. l_1), the lower right gray face represents lines having m_{pq} below them and a slope greater than the slope of (pq) (e.g. l_2).

We also have drawn the dual lines p^* and q^* of the sites and we can see that the dual line p^* is contained in the faces associated to p and does not cross faces associated to q.

Given now n sites, the line space Voronoi diagram of these sites can be thought as the intersection of all the $O(n^2)$ line space Voronoi diagrams for each pair of sites, which could result in a diagram of size $O(n^4)$. But in fact a lot of incident faces would represent lines closer to a same site and should be merged, so this brute-force approach is not interesting. We will therefore study directly the line space Voronoi diagram of n sites, show that its size is $O(n^2)$, and give an optimal algorithm to compute it explicitly.

3.2. Representation in dual space and combinatorics

From the previous section, we know that the supporting lines of the edges of the line space Voronoi diagram in dual space are the dual lines of the midpoints m_{pq} and the vertical dual lines of the parallels to two points (pq). We give more properties about faces and edges in the dual space.

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Property 6 *(1) The dual line p^* of a site p cuts all the faces associated to p and no other face.*

(2) The edges incident to a face associated to a site p have for supporting line either m^*_{pq} (the dual line of the midpoint of p and another site q) or $//^*_{pq}$ (the vertical dual line of parallels to a line passing through p and another site q). Moreover those supporting lines cannot cross any face associated to p. Therefore, an edge with supporting line m^*_{pq} or $//^*_{pq}$ separates a face associated to p and a face associated to q.

Proof: (1) A line passing through a site p is strictly closer to p so the dual line p^* belongs to the faces associated with p and a dual line q^* cannot cut faces associated to p.

(2) Consequence of the property 3 of equidistant lines. □

With these properties, we can draw the line space Voronoi diagram in the dual space: Figure 5 shows the line space Voronoi diagram of four sites. The dual lines of the midpoints and of the parallels to two sites are drawn thin-dashed, the dual lines of the sites are drawn thick-dashed, and the faces associated to a same site are drawn with the same level of gray.

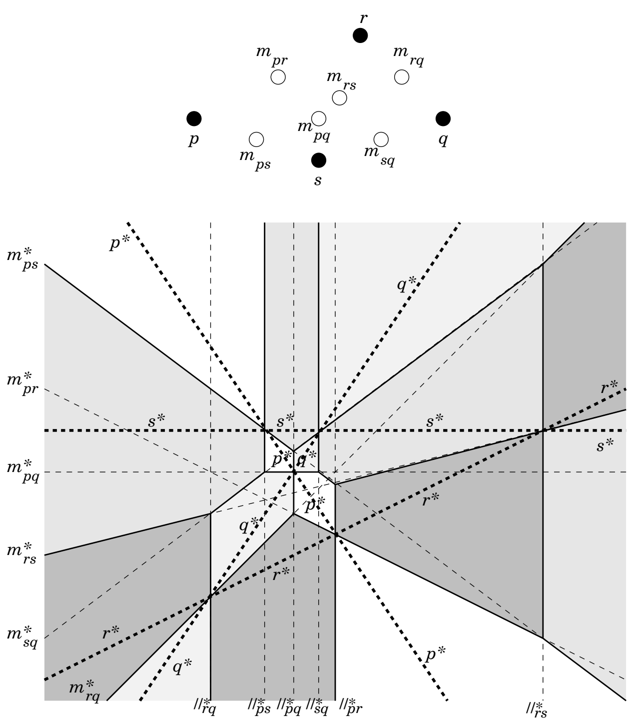

Figure 5. line space Voronoi diagram of four sites

Property 7 *In the dual space, the faces of the line space Voronoi diagram are convex.*

Proof: Let l_1^* and l_2^* be two points of a same face associated to p, let p_{12} be the intersection point of the corresponding lines l_1 and l_2, and let p_{sym} be the point symmetric of p according to p_{12}. The line segment $[l_1^*, l_2^*]$ corresponds to the set of lines passing through p_{12} and having a slope between those of l_1 and l_2. Since they belongs to a same face, l_1 and l_2 are both closer to p and have the same sites above and below (property 2). It follows that the double wedge between l_1 and l_2 is empty and that the strips delimited by the parallels to l_1 (resp. l_2) passing through p and p_{sym} are also empty. So the other sites are either above or below the four parallels (to l_1 and l_2 passing through p and p_{sym}) and for any line l passing through p_{12} in the double wedge of l_1 and l_2, the strip delimited by the parallels of l passing through p and p_{sym} is therefore empty (figures 6a and 6b). It shows that l is closer to p and have the same other sites below and above as l_1 and l_2, so l_1, l_2, and l belong to the same face. □

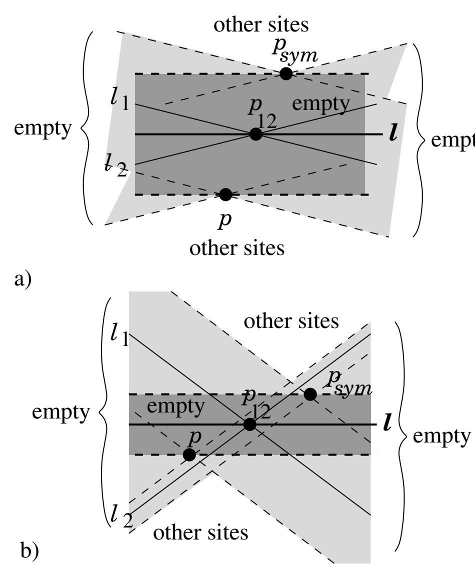

Figure 6. Configurations a) and b) of lines corresponding to a line segment of a face in the dual space

We now study more closely the vertices of this diagram, which will allow us to show that the size of the diagram is $\Theta(n^2)$ and to give an optimal algorithm for computing it. We saw that there are two kinds of vertices which correspond to lines (pq) and lines $(m_{pr}m_{qr})$. These vertices are not independent. First (pq) and $(m_{pr}m_{qr})$ are parallels, and since the strip delimited by (pq) and its parallel passing through r is empty, they are vertices incident to a same edge which corresponds to the parallels to (pq) between p (and therefore q) and r.

More generally, let p and q be two sites, and r and s the closest sites (other than p and q) to (pq) respectively above and below. Then we have the configuration shown in figure 7b in the dual space around the vertex corresponding to (pq). On the left (resp. right) figure 7a (resp. 7b) shows the sets of lines closer to a site for slopes smaller (resp. greater) than the slope of (pq). We can now clearly notice in the dual space that the vertices corresponding to a line passing through two sites are incident to four edges, and that the other vertices (corresponding to a line passing through two midpoints) are incident to three edges.

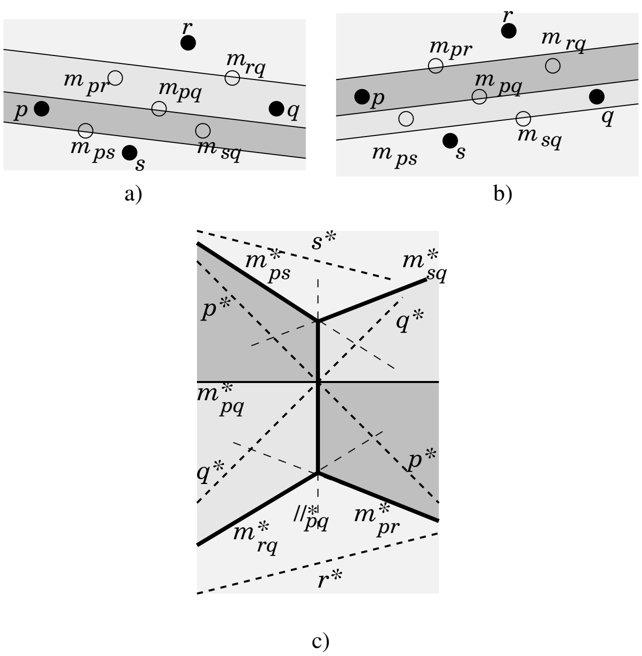

Figure 7. Configuration around a vertex of the line space Voronoi diagram a) b) in the scene c) in the dual space

We can now give the complexity of the line space Voronoi diagram:

Theorem 1 *The line space Voronoi diagram of n sites has $3n(n{-}1)/2-\text{CH}$ vertices, $n(5n-3)/2-1-\text{CH}$ edges, and n^2 faces (CH being the size of the convex hull of the sites).*

Proof: Each line (pq) passing through two sites is a vertex, and it always has another nearest site r on one side, which gives rise to another vertex passing through the midpoints m_{pr} and m_{qr}, and if $[pq]$ is not on the convex hull, then it has another nearest site on the other side which gives rise to another vertex, so the number of vertices is $n(n{-}1)/2+\text{CH}+2(n(n{-}1)/2-\text{CH})$.

There are n–1 semi-infinite edges on the left corresponding to midpoints of two consecutive sites when sorted according to their x-coordinate. When $[pq]$ is on the convex hull, it gives rise to 4 edges, otherwise to one more edge (i.e. 5 edges), so the number of edges is $n-1+4\,\text{CH}+5(n(n{-}1)/2-\text{CH})$.

There are n semi-infinite faces on the left. Each vertex (pq) gives rise to 2 faces, so the number of faces is $n+2n(n{-}1)/2$. □

We show now how to compute the line space Voronoi diagram with a topological sweep of its vertices.

The state of the sweep is represented by the cut of the diagram by a "flexible" vertical pseudo-line, that is, by the set of edges of the diagram cut by the vertical sweeping line. In the dual space, $-\infty$ corresponds to downward vertical lines, so the edges in the initial cut correspond to vertical lines passing through the midpoints of two consecutive sites according to their x-coordinate (and can be computed in $O(n \log n)$ time). Then we pass the vertices corresponding to lines passing through two sites in topological order (that is, if there is a strictly monotonous path in the diagram from v_1 to v_2, then v_1 must be passed before v_2). When we pass a vertex $v=(pq)$, thanks to the topological sweep, we have at hand the cut before the vertex, so we can compute the new vertices (v itself and the neighboring vertices on $//^*_{pq}$) and edges, and update the cut in constant time.

The vertices $v=(pq)$ of the line space Voronoi diagram correspond to the vertices $v=p^* \cap q^*$ of the dual arrangement of the sites, that is, the arrangement of the dual lines of the sites. A topological sweep of an arrangement of n lines can be done in optimal $O(n^2)$ time (see [4]), so the line space Voronoi diagram can be swept and computed explicitly also in optimal $O(n^2)$ time.

4 Relationship between the line space Voronoi diagram and the dual arrangement of the sites

The classical Voronoi diagram of a set of sites in the plane is a connected subset of the arrangement of the bisectors of the sites. We have here something somehow similar: in the dual space, the line space Voronoi diagram is a connected subset of the arrangement of the dual lines m^*_{pq} and $//^*_{pq}$, lines which are some kind of vertical and horizontal "bisectors" of the dual lines of the sites (any point on m^*_{pq} is vertically halfway between p^* and q^*, $//^*_{pq}$ is vertical and passes through the intersection point of p^* and q^*.)

We now shift our focus from the line space Voronoi diagram to the dual arrangement of the sites to show some remarkable relationships between these two data structures. It will allow us to show another way of computing the diagram and to prove a zone theorem in this diagram.

From the properties of the previous sections, we can state that:

Property 8 *A line segment of m^*_{pq} is an edge if and only if the first dual lines of a site above and below the line segment are p^* and q^*. Conversely, if p^* and q^* are two consecutive dual lines of sites in a vertical cut of the diagram, then the faces associated to p and q in this cut are separated by an edge whose supporting line is m^*_{pq}.*

With this property, we can compute the line space Voronoi diagram by transforming each face of the dual arrangement of the sites in the following way:

Property 9 *The line space Voronoi diagram can be computed by taking each face of the dual arrangement of the sites, making a vertical decomposition of the face according to its incident vertices, creating a vertex for each vertex of the arrangement and a vertex at the midpoint of each vertical segment of the decomposition, and creating an edge between vertices incident to a same trapeze and between vertices belonging to a same vertical segment (figure 8).*

Proof: For a trapeze whose upper and lower segment are supported by p^* and q^*, we can see that the supporting line of the edge linking the midpoints of the vertical segments of the trapeze is m^*_{pq}. □

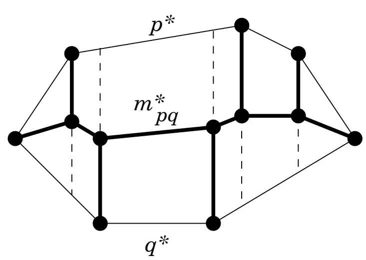

Figure 8. Computing the part of the line space Voronoi diagram contained in a face of the dual arrangement of the sites

We can now restate some of the properties we already have seen to sum up some properties between the line space Voronoi diagram and the dual arrangement of the sites:

Property 10 *A face f of the line space Voronoi diagram contains exactly one edge e of the dual arrangement of the sites and is cut into two by this edge. Moreover f is contained in the two faces of the dual arrangement incident to e.*

The part of the line space Voronoi diagram contained in a face of the dual arrangement of the sites of size (number of incident edges) f_a is made of:

- *f_a-2 vertices of degree 3, f_a vertices of degree 4, and $2f_a-3$ edges if the face is bounded,*
- *f_a-2 vertices of degree 3, f_a-1 vertices of degree 4, and $2f_a-3$ edges if the face is unbounded on the left or on the right,*
- *f_a-1 vertices of degree 4 and f_a-1 edges for the two vertically unbounded faces (on the top and on the bottom).*

By noticing that the sum of the sizes of the two vertically unbounded faces is equal to the size of the convex hull of the sites plus two, these properties give another way of computing the size of the line space Voronoi diagram.

4.1. New zone theorems

In an arrangement of n lines, the set $Z_0(l)$ of the faces of the arrangement cut by a line l is called the *zone of the line*, and the zone theorem (see e.g. [3] and [6]) states that the size of the zone (i.e., the sum of the sizes of its faces) of a line is in $O(n)$.

We give now a zone theorem for the zone of a line extended to its incident faces:

Theorem 2 *Given an arrangement of n lines, given a line l, let $Z_1(l)$ be the set of faces of the arrangement cut by l and of faces incident (by an edge) to a face cut by l. Then the sum of the sizes of the faces of $Z_1(l)$ is in $O(n)$.*

Proof: Our proof is based on the proof of the zone theorem given in [3].

We say that an edge is *connected* to $Z_0(l)$ if it is not incident to a face of $Z_0(l)$ but one of its incident vertex is. Since in a non-degenerate arrangement of n lines a vertex is incident to four edges, the number of edges connected to $Z_0(l)$ is in $O(n)$.

We count now the edges of $Z_1(l)$ located above l. Let $[a, b]$ be an edge with supporting line l_{ab}, a and b chosen such that a is nearer than b to the intersection point p_{ab} of l and l_{ab}. Now let l_a be the other line defining the vertex a and let p_a be the intersection point of l and l_a. The edge $[a, b]$ is called a *right-edge* (resp. *left-edge*) of l_a if p_a is on the right (resp. left) of p_{ab} (figure 9a). We show that each line h of the arrangement has at most two right-edges incident to a face of $Z_1(l)$ and not connected to $Z_0(l)$.

Let $[a, b]$, $[b, c]$, and $[x, y]$ be three right-edges of h, $[x, y]$ being above $[b, c]$ and $[b, c]$ being above $[a, b]$ (figure 9b). On the right of h, both lines l_{ab} and l_{cd} always hide $[x, y]$ from l, so $[x, y]$ cannot be incident to a face of $Z_1(l)$ that is on the right of h. On the left of h, h always hides $[x, y]$ from l. So the only way for $[x, y]$ to be incident to a face of $Z_1(l)$ that is on the left of h, is to be incident to a face incident to a face of the zone of l, which implies that x is a vertex incident to a face of the zone.

So for each line of the arrangement, there are at most two right-edges incident to a face of $Z_1(l)$ and not connected to $Z_0(l)$, and the total number of edges connected to $Z_0(l)$ is $O(n)$, so finally the size of $Z_1(l)$ is in $O(n)$. □

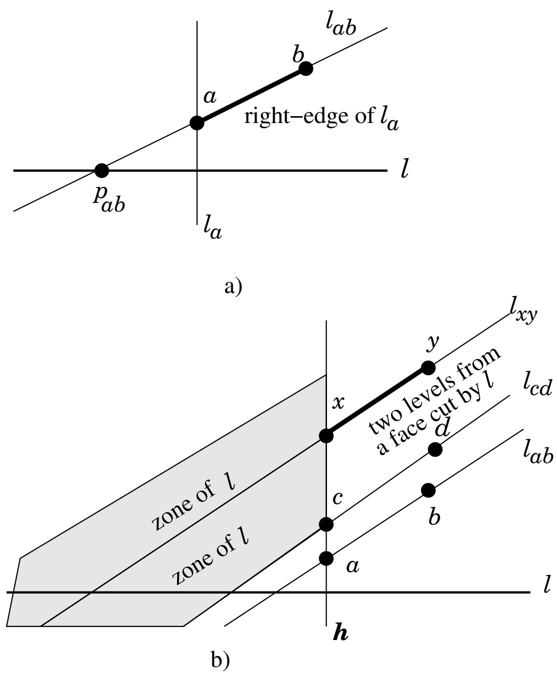

Figure 9. a) Right-edges of a line b) incident to a face of $Z_1(l)$

We can now state a zone theorem in the line space Voronoi diagram:

Property 11 *The size of the zone of a line in the line space Voronoi diagram is in $O(n)$.*

Proof: The faces of the zone of a line l in the line space Voronoi diagram are contained in the faces of $Z_1(l)$. Property 10 shows that the size of the part of the line space Voronoi diagram contained in a face of the dual arrangement is proportional to the size of the face. Since the size of $Z_1(l)$ is in $O(n)$, the size of the zone in the line space Voronoi diagram is also in $O(n)$. □

5. Conclusion

From the formal definition of the line space Voronoi diagram, we have given some properties, computed its size, and by using duality we have given an algorithm for computing it. Then we have shown that there is a remarkable close relationship between this diagram and the dual arrangement of lines of the sites.

The line space Voronoi diagram simplifies algorithms such that finding the site closer to a line in $O(\log n)$ time, finding the largest empty strip containing a given point or parallel to a given direction in $O(n)$ time, which could have been done by computing the vertical decomposition of the dual arrangement of the sites.

The relationships between the line space Voronoi diagram and the dual arrangement of the sites can allow us to extend more easily the line space Voronoi diagram to other types of sites than points, or to take visibility into account. In the first case, when taking convex objects as sites, the dual points of the tangents to an object give rise to two distinct curves in the dual space, so we can compute the arrangement of theses dual curves, and from its faces compute the line space Voronoi diagram. In the second case, when we want to take visibility into account, that is, we want to compute the set of maximal free line segments closer to a site, then instead of computing the dual arrangement, we can compute the visibility complex (see [8]), and try to apply the same decomposition process to its faces.

Finally, we are currently investigating the line space Voronoi diagram of order k and trying to see if there are still interesting relationships with the dual arrangement of the sites.

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