k-set polytopes and order-k Delaunay diagrams

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Abstract

Given a set S of n points (called sites) in a d-dimensional Euclidean space E and an integer k, $1 \leq k \leq n-1$, we consider three known structures that are defined through subsets of k elements of S: The k-set polytope of S, the order-k Voronoi diagram of S, and its dual, the order-k Delaunay diagram of S. We give a new compact characterization of all-dimensional faces of these three structures through the notions of k-couple and of k-set polytope of a k-couple. We also show that the incidence relations between these faces correspond to inclusion relations between k-couples. These characterizations allow us to give simple proofs of well known relations between the three structures, especially that the d-dimensional order-k Delaunay diagram is the projection of the lower hull of a $(d+1)$-dimensional k-set polytope and is the orthogonal dual of the order-k Voronoi diagram.

1. Introduction

Different problems that occur in computational geometry consist in studying subsets of k elements among n. For example, if S is a set of n sites in a d-dimensional Euclidean space E, the *k-sets* of S are the subsets of k sites of S that can be strictly separated from the remaining by a hyperplane (see figure 1). In the same way, the *order-k Voronoi diagram* of S is a partition of E whose regions are sets of points of E with the same k nearest neighbors in S.

A classical method in data analysis to express the distance from a point to a set of k sites consists in using the centroid (also called center of gravity) of these k sites. The centroid can also be used to reformulate geometric problems. In [10], we proved that the order-k Voronoi diagram admits a dual, called the *order-k Delaunay diagram*, whose vertices are the centroids of the subsets of k sites of S whose associated Voronoi regions are non-empty. As shown by Aurenhammer and Schwarzkopf [4], this dual diagram is the projection of a $(d+1)$-dimensional polyhedral convex surface. Similarly, Andrzejak and Fukuda [2] showed that finding the k-sets of S comes down to finding the vertices of the convex hull of the centroids of the k-element subsets of S. This convex hull is called the *k-set polytope* of S and has been introduced by Edelsbrunner, Valtr, and Welzl [6].

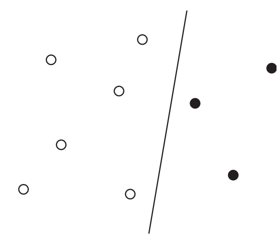

Figure 1. The black points form a 3-set of the whole set of points while the white points form a 6-set.

Thus, the knowledge of the complete structure of objects such as k-set polytopes and order-k Delaunay diagrams helps to find combinatorial and computational results for the underlying problems.

In [9], we studied the set of faces of the order-k Delaunay diagram and showed that each of its i-dimensional face ($i \in \{1, ..., d\}$) can be characterized by a sphere containing a subset P of S inside, passing through a subset Q, and such that $|P| < k < |P \cup Q|$. The property holds even in degenerate cases and allowed us to give an iterative construction algorithm for this diagram. In [1] and [3], Andrzejak and Welzl studied the faces of the k-set polytope of S when the sites are in general position. In this case, these faces are in fact the boundary faces of the order-k Delaunay diagram. They derived various linear relations among the number of these faces. Wagner [12] extended some of these results to the degenerate case where more than d sites may be coplanar. This occurs, for example, in k-set problems related to computational commutative algebra [7]. Wagner pointed

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out that every i-dimensional face ($i \in \{1, ..., d-1\}$) of such a k-set polytope is characterizable by an oriented hyperplane that passes through a subset Q of sites of S and has a subset P on its left such that $|P| < k < |P \cup Q|$.

It clearly appears that the characterization of order-k Delaunay diagrams and that of k-set polytopes are very close, the former involving separability by spheres and the latter by hyperplanes. This naturally calls for an unified characterization of the faces of these two structures, using separability by a surface. This is what we intend to do in this paper through the notion of k-*couple*, i.e., a couple (P, Q) of disjoint subsets of S for which either $|P| = k$ and $Q = \emptyset$, or $|P| < k < |P \cup Q|$ (see figure 2).

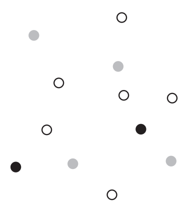

Figure 2. If S is the whole set of points, P the set of black points, and Q the set of grey points, then (P, Q) is a 3-couple of S (it is also a 4-couple and a 5-couple).

First, we define the new notion of k-set polytope of such a k-couple as being the convex hull of the centroids of the k-element subsets of $P \cup Q$ containing P and give some basic properties of these k-set polytopes. Then, we study the particular k-couples (P, Q) whose separating surface is a hyperplane, that is, the sites of Q lie in a hyperplane which separates P from $S \setminus (P \cup Q)$. Such k-couples are called *generalized* k-*sets* of S. We show that the k-set polytopes of these generalized k-sets are precisely the faces of the k-set polytope of S. We also show that the incidence relations between faces of the k-set polytope of S correspond to inclusion relations between generalized k-sets of S. This generalizes a result of [3] where the vertices of the k-set polytope faces have been characterized. Thanks to our definition of generalized k-sets, our relations hold for all-dimensional faces, including vertices. Moreover, all results are given in the general case where more than d sites may be coplanar.

Similarly, we consider the k-couples (P, Q) of S where the sites of Q belong to a sphere that separates the sites of P, inside the sphere, from the sites of $S \setminus (P \cup Q)$. Such k-couples are called k-*sections* of S. We show that the k-set polytopes of these k-sections form a partition of the k-set polytope of S. This partition is in fact the order-k Delaunay diagram of S defined in [9] and [10], in a completely different way, as the dual of the order-k Voronoi diagram. As for the k-set polytopes, we show that the incidence relations between all-dimensional order-k Delaunay faces can be interpreted as relations between k-sections. These later results also hold when any number of sites are cospherical. Moreover, we show that any face of the order-k Delaunay diagram is the projection of a face of a $(d+1)$-dimensional k-set polytope.

In the last section, we restate some of our results given in [11] but with shorter proofs based on the results of the previous sections. In particular, we give a bijection between the faces of the order-k Voronoi diagram of S and the k-sections of S which allows to prove easily the orthogonal duality between order-k Delaunay and Voronoi diagrams.

2. Generalized k-sets and k-set polytopes

Let S be a set of n sites in the Euclidean space E of dimension $dim(E) = d$ such that E is the affine hull $aff(S)$ of S and let k be an integer of $\{1, ..., n-1\}$. For every subset R of at least k sites of S, let ${}^k(R)$ be the set of k-element subsets of R and let $g^k(R)$ be the relative interior of the convex hull of the centroids of the elements of ${}^k(R)$, i.e., the greatest open subset of the affine hull of these centroids included in their convex hull. We call $g^k(R)$ the k-*set polytope* of R (see figure 3). For every oriented hyperplane π of E, we denote by π^+ the open half-space on the positive side of π and by π^- the other open half-space bounded by π. For every subset ω of E, we denote by $\overline{\omega}$ the smallest closed subset of E containing ω.

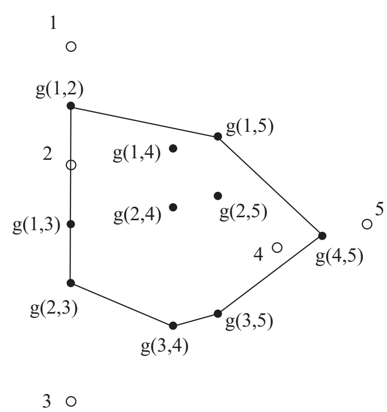

Figure 3. The centroids $g(s, t)$ of all pairs $\{s, t\}$ of $S = \{1, 2, 3, 4, 5\}$ and the 2-set polytope of S (1, 2, and 3 are collinear).

If T is a k-*set* of S, T is an element of ${}^k(S)$ for which there exists an oriented hyperplane π such that $\pi^+ \cap S = T$. T is then the only k-set of S that can be separated by an oriented hyperplane having the same outer normal as π. Thus, the set of directions of E determines the set of k-sets of S but the converse is not true. Indeed, if π is an oriented hyperplane such that $\pi \cap S = Q$ and $\pi^+ \cap S = P$ with $|P| < k < |P \cup Q|$, there is no hyperplane parallel to π and with the same orientation as π that determines a k-set of S (see figure 4). This leads to generalize the notion of k-set as follows:

- any couple (P, Q) of disjoint subsets of S such that $|P| = k$ if Q is empty and $|P| < k < |P \cup Q|$ otherwise, is called a *k-couple* of S
- moreover, if there exists a hyperplane containing Q and strictly separating P and $S \setminus (P \cup Q)$, then (P, Q) is called a *generalized k-set* of S.

Every oriented hyperplane π such that $\pi \cap S = Q$ and $\pi^+ \cap S = P$, is called a *separating hyperplane* of the generalized k-set (P, Q) in S; then every direction of E is the outer normal of separating hyperplanes of one and only one generalized k-set of S.

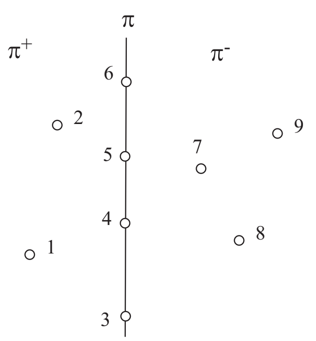

Figure 4. The couple $(\{1, 2\}, \{3, 4, 5, 6\})$ is a generalized k-set of $S = \{1, 2, ..., 9\}$ for every $k \in \{3, 4, 5\}$. π is here its unique separating line.

2.1. k-set polytopes and k-couples

Let us first study some basic properties of k-set polytopes and k-couples. For every subset T of S, let $g(T)$ be the centroid of T.

Lemma 1. *For every subset R of S with strictly more than k elements, the affine hulls $aff(g^k(R))$ and $aff(R)$ are equal.*

Proof. If G is the set of centroids of the elements of ${}^k(R)$, $aff(g^k(R)) = aff(G) \subseteq aff(R)$.

Let us suppose that $aff(G) \subset aff(R)$. Then, there exists an oriented hyperplane π of E that contains G but not R. Let T be a subset of k sites of R that is not included in π. Since $g(T) \in \pi$, there exist two sites s and t of T such that $s \in \pi^+$ and $t \in \pi^-$. Since $|T| < |R|$, there also exists a site r of $R \setminus T$ and r belongs to either the closed half-space $\overline{\pi^-}$ or $\overline{\pi^+}$. Hence the centroid $g((T \setminus \{s\}) \cup \{r\})$ belongs to π^- or π^+ respectively, which contradicts the hypothesis. It follows that $aff(G) = aff(g^k(R)) = aff(R)$. □

The property used in this proof can be generalized to give the following technical lemma (see figure 5 for an illustration).

Lemma 2. *Let T be a subset of k elements of S, U a non-empty subset of T, and π an oriented hyperplane of E such that $U \subset \overline{\pi^+}$. Let μ be the hyperplane parallel to π, with the same orientation as π, and that passes through $g(T)$.*

(i) For every subset V of $S \setminus T$ with same cardinality as U and included in $\overline{\pi^-}$, the centroid of $T' = (T \setminus U) \cup V$ belongs to $\overline{\mu^-}$.

(ii) Moreover, if at least one site of U belongs to π^+ or one site of V belongs to π^-, then $g(T') \in \mu^-$.

Proof. (i) Let Δ be a straight line orthogonal to π oriented from π^- to π^+ and let us consider the abscissae of the points of E on Δ. The abscissa of $g(T)$ on Δ is the average of the abscissae of the points of T on Δ. Since the abscissae of the points of V on Δ are smaller than or equal to the abscissae of the points of U on Δ, the average of the abscissae of the points of $T' = (T \setminus U) \cup V$ is smaller than or equal to the abscissae of $g(T)$. Thus $g(T')$ belongs to $\overline{\mu^-}$.

(ii) Moreover, if the abscissa of at least one point of V is strictly smaller than the abscissa of one point of U, the abscissa of $g(T')$ is strictly smaller than the abscissa of $g(T)$ and $g(T')$ belongs to μ^-. □

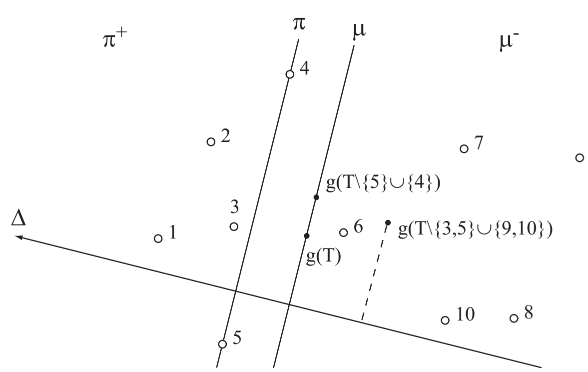

Figure 5. If $T = \{1, 2, 3, 5, 6, 7, 8\}$, $g(T)$ and $g(T \setminus \{5\} \cup \{4\})$ are on μ, but $g(T \setminus \{3, 5\} \cup \{9, 10\})$ belongs to μ^-.

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For every k-couple (P, Q) of S, we denote by $^k(P, Q)$ the set of subsets of k elements of $P \cup Q$ that contain P. Equivalently, $^k(P, Q)$ is the set of all subsets of $^k(S)$ whose intersection is P and whose union is $P \cup Q$.

It follows that $(P, Q) \neq (P', Q')$ implies $^k(P, Q) \neq {}^k(P', Q')$.

The relative interior of the convex hull of the centroids of the elements of $^k(P, Q)$ is called the *k-set polytope* of (P, Q) and is denoted by $g_P^k(Q)$ (see figure 6).

Note that the k-set polytope of $(\emptyset, Q)$ is nothing else but the k-set polytope of Q.

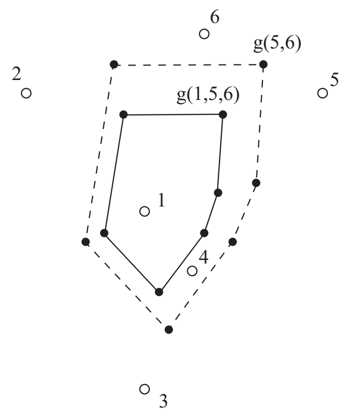

Figure 6. The 3-set polytope $g_1^3(2,3,4,5,6)$ of $(\{1\},\{2,3,4,5,6\})$ (solid lines) is the image of the 2-set polytope $g^2(2,3,4,5,6)$ of $\{2,3,4,5,6\}$ (dashed lines) by the homothety of ratio $2/3$ centered at site 1.

Lemma 3. *For every k-couple (P, Q) of S, $dim(g_P^k(Q)) = dim(Q)$ and, if Q is not empty, $g_P^k(Q)$ is parallel to $aff(Q)$.*

Proof. If Q is not empty, $dim(Q) \neq 0$ by the definition of k-couples. Moreover, if P is empty $g_P^k(Q) = g^k(Q)$ and otherwise $g_P^k(Q)$ is the image of $g^{k-|P|}(Q)$ by the homothety of ratio $(k-|P|)/k$ centered at $g(P)$. Since, by lemma 1, $aff(Q) = aff(g^{k-|P|}(Q))$, $g_P^k(Q)$ and Q have the same dimension and are parallel.

If Q is empty, $^k(P, \emptyset) = \{P\}$, $g_P^k(\emptyset)$ is reduced to the centroid of P, and $dim(Q) = dim(g_P^k(Q))$ holds true. $\square$

2.2. Generalized k-sets

Now we show that the k-set polytopes of the generalized k-sets of S are the faces of the k-set polytope of S (see figure 7).

By "face of a polytope" of E we mean any i-dimensional face of this polytope, $i \in \{0, ..., d-1\}$. The faces of dimension 0, 1, and $d-1$ are also respectively called vertices, edges, and facets of the polytope.

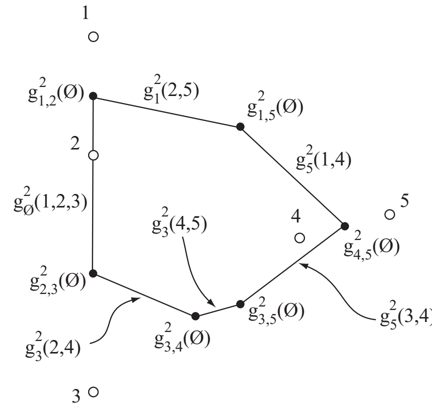

Figure 7. The 2-set polytopes of the generalized 2-sets of $S = \{1, 2, 3, 4, 5\}$ are the faces (vertices and edges) of the 2-set polytope of S.

Lemma 4. *(i) For every generalized k-set (P, Q) of S, $g_P^k(Q)$ is a face of $g^k(S)$.*

(ii) The k-set polytopes of the generalized k-sets of S are pairwise disjoint.

Proof. (i) If π is a separating hyperplane of a generalized k-set (P, Q) of S, by lemma 3 there exists a hyperplane μ parallel to π, with the same orientation as π, and that contains $g_P^k(Q)$.

By definition, for every element T of $^k(S) \setminus {}^k(P, Q)$, either T does not contain at least one site of $P = \pi^+ \cap S$, or T does contain at least one site of $S \setminus (P \cup Q) = \pi^- \cap S$. Thus, by lemma 2, $g(T)$ belongs to the open half-space μ^- and μ is a supporting hyperplane of $g^k(S)$ such that $\overline{g_P^k(Q)} = \mu \cap \overline{g^k(S)}$. Therefore $g_P^k(Q)$ is a face of $g^k(S)$.

(ii) It follows from (i) that the k-set polytopes of the generalized k-sets of S are either disjoint or identical. Now, if (P', Q') is a generalized k-set of S distinct from (P, Q), then $^k(P', Q') \neq {}^k(P, Q)$. From the proof of (i), if there exists $T \in {}^k(P', Q') \setminus {}^k(P, Q)$, then $g(T)$ belongs to μ^- and $g_{P'}^k(Q') \neq g_P^k(Q)$. By symmetry, it follows that the k-set polytopes of the generalized k-sets of S are pairwise disjoint. $\square$

Theorem 1. *The k-set polytopes of the generalized k-sets of S are pairwise disjoint and are the faces of the k-set polytope of S.*

Proof. (i) From lemma 4, the k-set polytopes of the generalized k-sets of S are pairwise disjoint and every such k-set polytope is a face of the k-set polytope of S. Thus, it remains to prove that each face of $g^k(S)$ is the k-set polytope of a generalized k-set of S.

(ii) For every face f of $g^k(S)$, let $\mathcal{T} = \{T_1, T_2, ..., T_m\}$ be the set of elements of ${}^k(S)$ whose centroids lie in $\overline{f}$. Let μ be a supporting hyperplane of $g^k(S)$ such that $\overline{g^k(S)} \cap \mu = \overline{f}$ and suppose that μ is oriented in such a way that $g^k(S)$ is included in μ^-. Let π be the hyperplane parallel to μ, oriented as μ, passing through a site t of $V = T_1 \cup T_2 \cup ... \cup T_m$, and such that all sites of V belong to $\overline{\pi^+}$ (see figure 8).

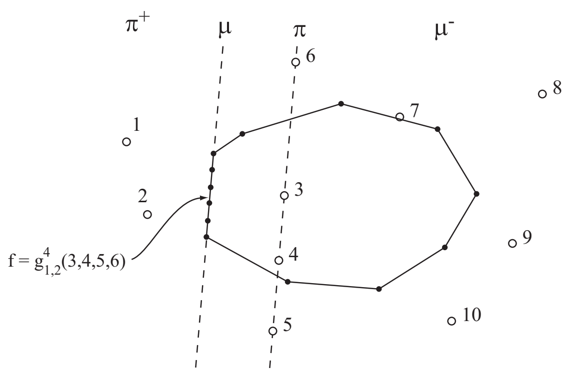

Figure 8. The 4-tuples whose centroids belong to the closed edge $\overline{f}$ of this 4-set polytope are of the form $\{1, 2, p, q\}$ with $\{p, q\} \subset \{3, 4, 5, 6\}$.

No site s of $S \setminus V$ can belong to $\overline{\pi^+}$ since otherwise, for every element T_i of $\mathcal{T}$ containing t, the centroid $g((T_i \setminus \{t\}) \cup \{s\})$ would belong to $\overline{\mu^+}$ by lemma 2 and $(T_i \setminus \{t\}) \cup \{s\}$ would belong to $\mathcal{T}$. This is in contradiction with the hypothesis that $s \notin T_1 \cup T_2 \cup ... \cup T_m$ and that $\mathcal{T} = \{T_1, T_2, ..., T_m\}$. Hence, $V = S \cap \overline{\pi^+}$. Let us now distinguish the two cases $|\mathcal{T}| = 1$ and $|\mathcal{T}| > 1$.

(ii.1) If $\mathcal{T}$ is reduced to the single element T_1, then $V = T_1$ is a k-set of S, ${}^k(T_1, \emptyset) = \{T_1\}$, and $f = g^k_{T_1}(\emptyset) = g(T_1)$ is a vertex of $g^k(S)$.

(ii.2) If $\mathcal{T}$ contains at least two elements, $|V| > k$ and $P = S \cap \pi^+$ cannot contain k elements since otherwise its centroid would belong to μ^+. It follows that $|P| < k$. Thus, setting $Q = V \setminus P = S \cap \pi$, (P, Q) is a generalized k-set of S with π as a separating hyperplane.

Let us now show that $\mathcal{T} = {}^k(P, Q)$. If there would exist an element T_i of $\mathcal{T}$ not containing P, then for every site s of $P \setminus T_i$ and for every site t of $T_i \cap Q = T_i \cap \pi$, $g((T_i \setminus \{t\}) \cup \{s\})$ would belong to μ^+ by lemma 2, in contradiction with the fact that $g^k(S) \subset \mu^-$. Hence, for every $i \in \{1, ..., m\}$, $P \subset T_i$ and, since $T_i \subset V = P \cup Q$, $\mathcal{T} \subseteq {}^k(P, Q)$. Thus $g^k_P(Q) \cap \mu$ contains $\{g(T_1), ..., g(T_m)\}$ and $g^k_P(Q) \cap \mu \neq \emptyset$. Since $g^k_P(Q)$ is parallel to Q by lemma 3 and since Q is included in π, $g^k_P(Q)$ is also parallel to μ. It follows that $g^k_P(Q) \subset \mu$ and that ${}^k(P, Q) \subseteq \mathcal{T}$. Therefore, $\mathcal{T} = {}^k(P, Q)$ and $g^k_P(Q) = f$.

(iii) Finally, it follows from (i) and (ii) that the k-set polytopes of the generalized k-sets of S are the faces of $g^k(S)$ and are disjoint. □

Let us now study the adjacency relations between faces of the k-set polytope of S.

Theorem 2. *For every generalized k-set (P, Q) of S such that $dim(Q) > 0$, the faces of $g^k_P(Q)$ are the k-set polytopes of the couples $(P \cup P', Q')$ that verify the following equivalent properties:*

(1) (P', Q') is a generalized $(k - |P|)$-set of Q.

(2) $(P \cup P', Q')$ is a generalized k-set of S distinct from (P, Q) such that $P' \cup Q' \subseteq Q$.

Proof. (1) If P is empty, $g^k_P(Q) = g^k(Q)$ is the k-set polytope of Q and, by theorem 1, its faces are the k-set polytopes of the generalized k-sets of Q.

If P is not empty, $g^k_P(Q)$ is the image of $g^{k-|P|}_{\emptyset}(Q) = g^{k-|P|}(Q)$ by the homothety $\mathcal{H}$ of ratio $(k - |P|)/k$ centered at $g(P)$. Now, by theorem 1, the faces of $g^{k-|P|}(Q)$ are the $(k - |P|)$-set polytopes $g^{k-|P|}_{P'}(Q')$ of the generalized $(k - |P|)$-sets (P', Q') of Q. Since the homothety $\mathcal{H}$ maps the faces of $g^{k-|P|}(Q)$ into those of $g^k_P(Q)$ and also maps $g^{k-|P|}_{P'}(Q')$ into the k-set polytope $g^k_{P \cup P'}(Q')$ of the k-couple $(P \cup P', Q')$, it follows that the faces of $g^k_P(Q)$ are the k-set polytopes of the couples $(P \cup P', Q')$ such that (P', Q') is a generalized $(k - |P|)$-set of Q.

(2) From (1), every face of $g^k_P(Q)$ is of the form $g^k_{P \cup P'}(Q')$ with $P' \cup Q' \subseteq Q$. Furthermore, if π is a separating hyperplane of (P, Q) in S and π' is a separating hyperplane of (P', Q') in Q, we get a separating hyperplane π'' of $(P \cup P', Q')$ in S by rotating π around $\pi \cap \pi'$ (see figure 9). Thus, $(P \cup P', Q')$ is a generalized k-set of S.

Conversely, if $(P \cup P', Q')$ is a generalized k-set of S distinct from (P, Q) such that $P' \cup Q' \subseteq Q$ then, for every element T' of ${}^k(P \cup P', Q')$, $P \subseteq P \cup P' \subseteq T' \subseteq P \cup P' \cup Q' \subseteq P \cup Q$. Thus T' also belongs to ${}^k(P, Q)$ and $g^k_{P \cup P'}(Q') \subseteq \overline{g^k_P(Q)}$. Since the k-set polytopes of the generalized k-sets of S are pairwise disjoint and since their faces are the k-set polytopes of the generalized k-sets of S, it follows that $g^k_{P \cup P'}(Q')$ is a face of $g^k_P(Q)$ (see figure 10). □

Remark 1. *More generally, property (1) of theorem 2 holds for every k-couple (P, Q) of S since its proof does not use the fact that (P, Q) is a generalized k-set of S.*

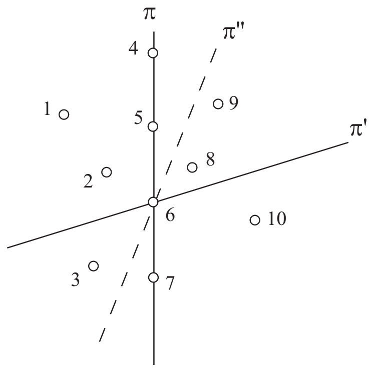

Figure 9. π is a separating straight line of $(\{1,2,3\},\{4,5,6,7\})$ in $S=\{1,...,10\}$ and π' is a separating straight line of $(\{4,5\},\{6\})$ in $\{4,5,6,7\}$. By an appropriatly slight rotation of π around $\{6\}=\pi\cap\pi'$, we get a straight line π'' which is separating for $(\{1,2,3,4,5\},\{6\})$ in S.

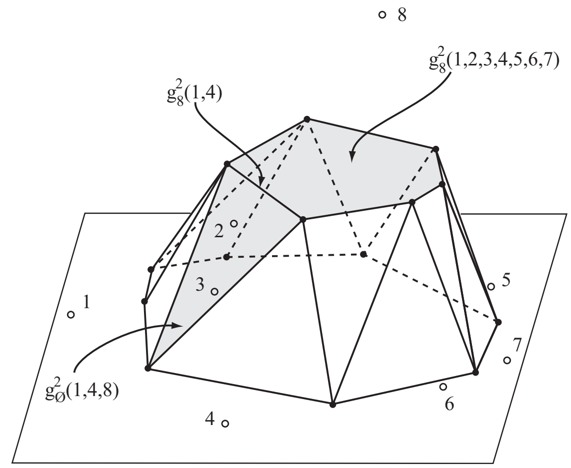

Figure 10. The 2-set polytope of 8 sites in dimension 3 with $\{1,2,...,7\}$ coplanar. The grey faces $g_8^2(1,2,...,7)$ and $g_\emptyset^2(1,4,8)$ share $g_8^2(1,4)$ as a common edge. Its endpoints are the vertices $g_{1,8}^2(\emptyset)$ and $g_{4,8}^2(\emptyset)$.

3. k-sections and order-k Delaunay diagram

In the previous section, we studied the k-set polytopes of the k-couples (P,Q) when the sites of Q lay on a separating hyperplane. Since the notion of k-set polytope is defined for any k-couple of S, the previous study can be extended to other kinds of k-set polytopes. In this section, we will consider the k-set polytopes of the k-couples (P,Q) for which the sites of Q lie on a separating sphere.

3.1. Order-k Delaunay diagram

For every sphere σ of E, let σ^+ and σ^- be the open subsets of E respectively inside and outside σ. A k-couple (P,Q) of S for which there exists a sphere σ such that $\sigma\cap S=Q$ and $\sigma^+\cap S=P$, is called a *k-section* of S and σ is called a *separating sphere* of the k-section (P,Q) in S.

In the special case where $k=1$, the 1-set polytopes of the 1-sections of S are the polytopes, with vertices in S, that are inscribable in empty spheres (i.e., that contain no site of S inside). Thus, these 1-set polytopes are the faces of the Delaunay diagram of S. More generally, the k-set polytopes of the k-sections of S will be called the *order-k Delaunay faces* of S.

In this section we will show that the set of order-k Delaunay faces of S forms a partition of $\overline{g^k(S)}$ called the *order-k Delaunay diagram* of S (see figure 11).

By "face of a diagram" of E we mean any i-dimensional face of this diagram, $i\in\{0,...,d\}$. The d-dimensional faces are also called regions of the diagram.

Theorem 3. *The order-k Delaunay faces of S are pairwise disjoint.*

Proof. (i) Let (P,Q) and (P',Q') be two distinct k-sections with σ and σ' as respective separating spheres.

Let us first show that none of the spheres σ and σ' can be inside the other. In order to prove it by contradiction we may assume, without loss of generality, that σ' is included in $\overline{\sigma^+}$.

If no site of S coincides with the touching point $\sigma\cap\sigma'$ (if it exists), then $P'\cup Q'$ is included in σ^+ and therefore in P. Since (P,Q) and (P',Q') are k-couples, it follows that $|P'\cup Q'|=|P|=k$ and that $P'\cup Q'=P$. This implies that $Q'=\emptyset$, $P'=P$, and $Q=\emptyset=Q'$, in contradiction with $(P,Q)\neq(P',Q')$.

If σ' is inwardly tangent to σ and if $\sigma\cap\sigma'$ contains a site q of $Q\cap Q'$, $P'\cup(Q'\setminus\{q\})$ is included in σ^+ and therefore in P. This is impossible since $|P'\cup Q'|>k$ and $|P|<k$ in the case where Q and Q' are non-empty.

(ii) When σ^+ and σ'^+ are disjoint, $g^k_P(Q)$ and $g^k_{P'}(Q')$ are also disjoint since $g^k_P(Q)\subset\sigma^+$ and $g^k_{P'}(Q')\subset\sigma'^+$.

(iii) When σ and σ' intersect without being tangent, let π be their radical hyperplane oriented in such a way that $\sigma\cap\sigma'^-\subset\pi^-$ (see figure 12). Thus $\overline{\sigma^+}\cap\pi^+\subset\sigma'^+$ and $\overline{\sigma'^+}\cap\pi^-\subset\sigma^+$. Therefore $(P\cup Q)\cap\pi^+\subseteq P'$ and $(P'\cup Q')\cap\pi^-\subseteq P$. Let μ be the supporting hyperplane of $g^k_P(Q)$ that is parallel to π, oriented like π, and such that $\overline{g^k_P(Q)}$ is included in $\overline{\mu^-}$. Then there exists at least one

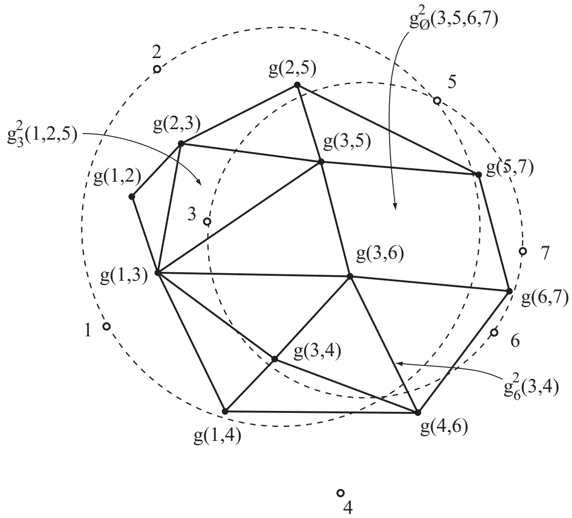

Figure 11. The 2-set polytopes of the 2-sections of $S = \{1, 2, ..., 7\}$ form the order-2 Delaunay diagram of S. Its vertices are the centroids of the couples of circularly separable sites of S.

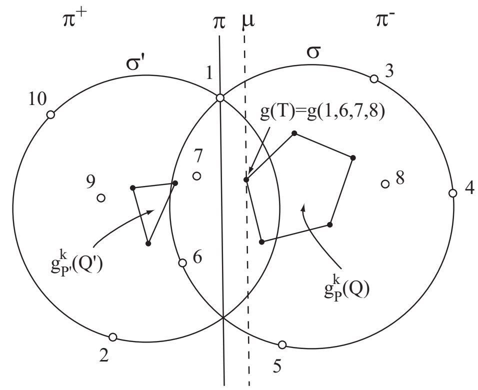

Figure 12. The 4-set polytopes $g^k_{P'}(Q') = g^4_{6,7,9}(1,2,10)$ and $g^k_P(Q) = g^4_{7,8}(1,3,4,5,6)$ are on both sides of the supporting straight line μ of $g^k_P(Q)$.

element T of $^k(P,Q)$ such that $g(T)$ is a point of $\overline{g^k_P(Q)} \cap \mu$. For every T' of $^k(P',Q')$, $U = T \setminus T' \subset \overline{\pi^-}$ since $T \cap \pi^+ \subset (P \cup Q) \cap \pi^+ \subseteq P' \subseteq T'$. Symmetrically, $V = T' \setminus T \subset \overline{\pi^+}$. By lemma 2, $g(T') = g((T \setminus U) \cup V) \in \overline{\mu^+}$. Hence $\overline{g^k_{P'}(Q')} \subset \overline{\mu^+}$ and $\overline{g^k_P(Q)} \cap \overline{g^k_{P'}(Q')} \subset \mu$, since $\overline{g^k_P(Q)} \subset \overline{\mu^-}$.

Case 1. Since $g^k_P(Q)$ is open, if $\overline{g^k_P(Q)}$ intersects the open half-space μ^-, then μ is a supporting hyperplane of $g^k_P(Q)$ and does not intersect $g^k_P(Q)$. Thus, $g^k_P(Q)$ is included in μ^-, and $g^k_{P'}(Q')$ and $g^k_P(Q)$ are disjoint.

It is the same when $\overline{g^k_{P'}(Q')} \cap \mu^+ \neq \emptyset$ and we are left with the cases where both $g^k_P(Q)$ and $g^k_{P'}(Q')$ are included in μ.

Case 2. If $g^k_P(Q) \cup g^k_{P'}(Q') \subset \mu$ and $P \neq P'$ then, within a permutation of P and P', there exists $p \in P' \setminus P$. Since $P' \setminus P$ is included in both π^+ and $T' \setminus T = V$, p belongs to both V and π^+. By lemma 2, it follows that $g(T') = g((T \setminus U) \cup V) \in \mu^+$, a contradiction.

Case 3. Let us now deal with the case $g^k_P(Q) \cup g^k_{P'}(Q') \subset \mu$ and $P = P'$. Since $aff(Q)$ and $aff(Q')$ are respectively parallel to $g^k_P(Q)$ and $g^k_{P'}(Q')$ by lemma 3, there exist two hyperplanes ξ and ξ' parallel to μ that contain Q and Q' respectively. Now, since $Q \cap \pi^+ \subseteq P'$ and $P \cap Q = \emptyset$, $P = P'$ implies $Q \cap \pi^+ = \emptyset$ and consequently $Q \subset \overline{\pi^-}$. Similarly, $Q' \subset \overline{\pi^+}$ and, since $Q \cap \pi = Q' \cap \pi$ and $Q \neq Q'$, Q and Q' cannot all together be included in π. Hence $\xi \neq \xi'$. If P and P' are empty, $g^k_P(Q)$ and $g^k_{P'}(Q')$ are included in ξ and ξ', and otherwise they are included in two parallel hyperplanes which are images of ξ and ξ' by the same homothety of center $g(P) = g(P')$. Therefore, in all cases, the hyperplanes parallel to μ that contain $g^k_P(Q)$ and $g^k_{P'}(Q')$ are distinct, in contradiction with the hypothesis.

It follows that $g^k_P(Q)$ and $g^k_{P'}(Q')$ cannot all together be included in μ. Hence they are disjoint. □

As for generalized k-sets, the following theorem gives the adjacency relations between order-k Delaunay faces of S.

Theorem 4. *For every k-section (P,Q) of S such that $dim(Q) > 0$, the faces of $g^k_P(Q)$ are the k-set polytopes of the couples $(P \cup P', Q')$ that verify the following equivalent properties:*

(1) *(P', Q') is a generalized $(k - |P|)$-set of Q.*

(2) *(P', Q') is a $(k - |P|)$-section of Q.*

(3) *$(P \cup P', Q')$ is a k-section of S distinct from (P, Q) such that $P' \cup Q' \subseteq Q$.*

Proof. (i) From remark 1, every face of $g^k_P(Q)$ is the k-set polytope of a k-couple $(P \cup P', Q')$ such that (P', Q') is a generalized $(k - |P|)$-set of Q. This shows (1).

(ii) Let us show that (1) implies (2).

If π is a separating hyperplane of the generalized $(k - |P|)$-set (P', Q') in Q, there exists a sphere σ' such that $\sigma' \cap Q = \pi \cap Q = Q'$ and $\sigma'^+ \cap Q = \pi^+ \cap Q = P'$. σ' is then a separating sphere of (P', Q') in Q and (P', Q') is a $(k - |P|)$-section of Q.

(iii) Let us now show that (2) implies (3).

For every $(k-|P|)$-section (P',Q') of Q, $P'\cup Q'\subseteq Q$. Moreover, if σ is a separating sphere of (P,Q) in S with center x and σ' a separating sphere of (P',Q') in Q with center x', there exists a sphere σ'' passing through $\sigma\cap\sigma'$, whose center x'' is close to x on $]x,x'[$, and that is separating for $(P\cup P',Q')$ in S (see figure 13). Thus, $(P\cup P',Q')$ is a k-section of S.

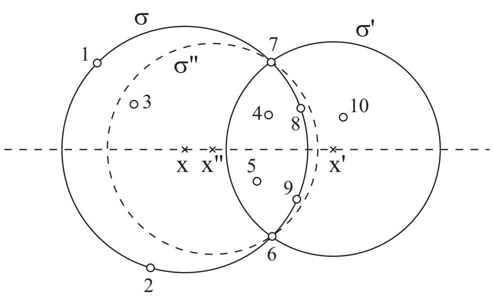

Figure 13. Since the couple $(P,Q) = (\{3,4,5\},\{1,2,6,7,8,9\})$ is a 6-section of $S=\{1,...,10\}$ and the couple $(P',Q') = (\{8,9\},\{6,7\})$ is a 3-section of Q, the couple $(P\cup P',Q') = (\{3,4,5,8,9\},\{6,7\})$ is a 6-section of S.

(iv) Let us finally show that the k-set polytope of a couple that verifies property (3) is a face of $g_P^k(Q)$.

If $(P\cup P',Q')$ is a k-section of S distinct from (P,Q) such that $P'\cup Q'\subseteq Q$, then for every T' of ${}^k(P\cup P',Q')$, $P\subseteq P\cup P'\subseteq T'\subseteq P\cup P'\cup Q'\subseteq P\cup Q$. It follows that T' belongs to ${}^k(P,Q)$ and that $g^k_{P\cup P'}(Q')\subseteq \overline{g^k_P(Q)}$. Since the order-$k$ Delaunay faces of S are pairwise disjoint by theorem 3 and since their faces are order-k Delaunay faces, $g^k_{P\cup P'}(Q')$ is a face of $g^k_P(Q)$ (see figure 14). ☐

A *k-section* (P,Q) of S is said to be *unbounded*, if its separating spheres can have arbitrarily large radii. Note that the notions of unbounded k-section and generalized k-set are not equivalent in degenerate cases, i.e. if $j+2$ sites lie on a common j-dimensional plane.

Lemma 5. *(i) For every unbounded k-section (P,Q) of S such that $dim(Q)=d-1$, there is a unique k-section (P',Q') such that $g^k_P(Q)$ is a facet of $g^k_{P'}(Q')$. Moreover, $g^k_P(Q)$ is included in the boundary of $g^k(S)$.*

(ii) For every bounded k-section (P,Q) of S such that $dim(Q)=d-1$, there are exactly two k-sections (P',Q') and (P'',Q'') such that $g^k_P(Q)$ is a facet of $g^k_{P'}(Q')$ and of $g^k_{P''}(Q'')$. Moreover, $g^k_{P'}(Q')$ and $g^k_{P''}(Q'')$ are on both sides of $g^k_P(Q)$.

Proof. (i) Let σ be a separating sphere of the k-section (P,Q) and let π be the hyperplane $aff(Q)$ with an arbitrary

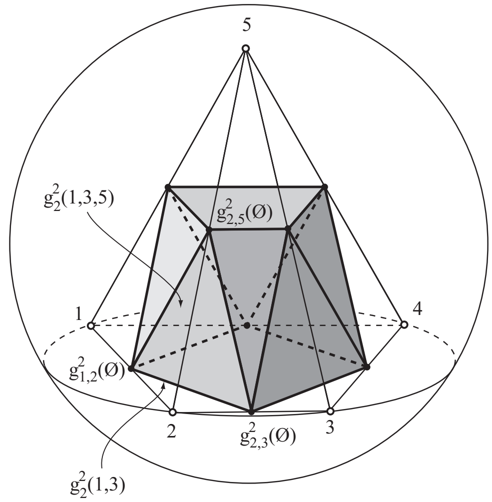

Figure 14. 1-set polytope (thin lines) and 2-set polytope (thick lines) of 2-section $(\emptyset,\{1,2,3,4,5\})$ in dimension 3. $g^2_{2,5}(\emptyset)$, $g^2_{1,2}(\emptyset)$, and $g^2_{2,3}(\emptyset)$ are the vertices of face $g^2_2(1,3,5)$ and $g^2_2(1,3)$ is one of its edges.

orientation. Then there exist two separating spheres σ' and σ'' of (P,Q) such that $\sigma'\cap\sigma^-\subset\pi^+$ and $\sigma''\cap\sigma^-\subset\pi^-$.

(i.1) If (P,Q) is unbounded we can suppose, without loss of generality, that the radius of σ' can be arbitrarily large (see figure 15). Therefore, $P\subset\overline{\pi^+}$ and $S\setminus(P\cup Q)\subset\overline{\pi^-}$. Setting $Q'=S\cap\pi$ and $P'=S\cap\pi^+=P\setminus\pi$, we get $|P'|\leq|P|<k$ and $|P'\cup Q'|\geq|P\cup Q|>k$. (P',Q') is thus a generalized k-set of S of separating hyperplane π and, by theorem 1, $g^k_{P'}(Q')$ is a facet of $g^k(S)$. Furthermore, since $P'\subseteq P$ and $P\cup Q\subseteq P'\cup Q'$, we have ${}^k(P,Q)\subseteq {}^k(P',Q')$. Hence $g^k_P(Q)$ is included in $\overline{g^k_{P'}(Q')}$, and therefore in the boundary of $g^k(S)$.

(i.2) If the radius of σ'' could also be arbitrarily large, we would have $P\subset\overline{\pi^-}$ and $S\setminus(P\cup Q)\subset\overline{\pi^+}$, that is $S\subset\pi$, from (i.1). This is impossible since $aff(S)=E$ by hypothesis. Thus there exists a limit sphere σ''_l which passes through the sites of Q and at least one site of $S\setminus Q$, and such that $P\subset\overline{\sigma''^+_l}$ and $S\setminus(P\cup Q)\subset\overline{\sigma''^-_l}$.

Setting $P''=\sigma''^+_l\cap S$ and $Q''=\sigma''_l\cap S$, we have $(P'',Q'')\neq(P,Q)$, $P''\subseteq P$, and $P\cup Q\subseteq P''\cup Q''$. Thus (P'',Q'') is a k-section of S of separating sphere σ''_l and, by theorem 4, $g^k_P(Q)$ is a facet of the order-k Delaunay region $g^k_{P''}(Q'')$. Since the order-k Delaunay regions are pairwise disjoint and since they are included in $\overline{g^k(S)}$, $g^k_{P''}(Q'')$ is the only order-k Delaunay region from which $g^k_P(Q)$ is a facet.

(ii) If (P,Q) is a bounded k-section, there exist two limit

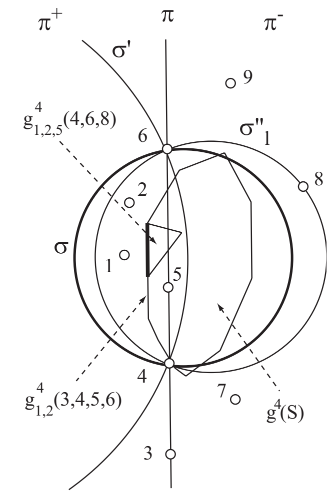

Figure 15. The separating circle σ' of the unbounded 4-section $(\{1, 2, 5\}, \{4, 6\})$ can tend toward the separating line π of the generalized 4-set $(\{1, 2\}, \{3, 4, 5, 6\})$. Thus, $g^4_{1,2,5}(4, 6)$ (thick) is included in the edge $g^4_{1,2}(3, 4, 5, 6)$ of the 4-set polytope $g^4(S)$ of $S = \{1, 2, ..., 9\}$. $g^4_{1,2,5}(4, 6)$ is also an edge of the order-4 Delaunay region $g^4_{1,2,5}(4, 6, 8)$.

spheres σ'_l and σ''_l (see figure 16). Setting $P' = \sigma'^+_l \cap S$, $Q' = \sigma'_l \cap S$, $P'' = \sigma''^+_l \cap S$, and $Q'' = \sigma''_l \cap S$, (P', Q') and (P'', Q'') are k-sections of S, and the two order-k Delaunay regions $g^k_{P'}(Q')$ and $g^k_{P''}(Q'')$ share $g^k_P(Q)$ as a common facet. Since the order-k Delaunay regions of S are disjoint, $g^k_{P'}(Q')$ and $g^k_{P''}(Q'')$ are on both sides of $g^k_P(Q)$ and are unique. □

Theorem 5. *(i) The order-k Delaunay faces of S form a partition of $\overline{g^k(S)}$.*

(ii) The k-set polytopes of the unbounded k-sections of S form a partition of the boundary of $g^k(S)$.

Proof. (i) Since $|S| > k$, there exists at least one k-section (P, Q) with $dim(Q) = dim(S) = d$. $g^k_P(Q)$ is then an order-k Delaunay region of S. By theorem 4, every facet of $g^k_P(Q)$ is an order-k Delaunay facet of S and, by lemma 5, this facet is included either in the boundary of $g^k(S)$ or in the boundary of another order-k Delaunay region. Thus, the set of closed order-k Delaunay regions covers $\overline{g^k(S)}$. Moreover, the faces of the order-k Delaunay regions are the order-k Delaunay faces by theorem 4 and are pairwise disjoint by theorem 3. Hence, the order-k Delaunay faces form a partition of $\overline{g^k(S)}$ (see figure 17).

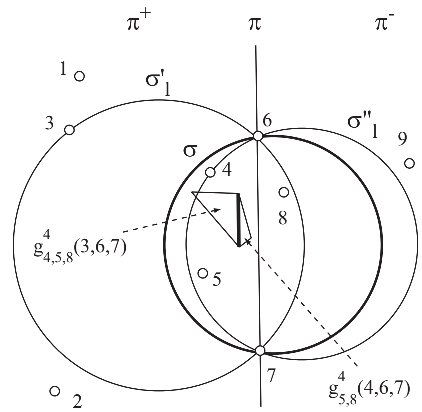

Figure 16. The 4-set polytope $g^4_{4,5,8}(6, 7)$ (thick segment) of the bounded 4-section $(\{4, 5, 8\}, \{6, 7\})$ is the common edge of the two order-4 Delaunay regions $g^4_{4,5,8}(3, 6, 7)$ and $g^4_{5,8}(4, 6, 7)$.

(ii) From (i), the boundary of $g^k(S)$ is split up into disjoint order-k Delaunay faces of S. Moreover, by lemma 5, their defining k-sections are the unbounded k-sections of S. □

In case the sites of S are in general position, the unbounded k-sections of S are the generalized k-sets of S. Thus, by theorems 1 and 5, in this special case the faces of the boundary of the order-k Delaunay diagram of S are the faces of the k-set polytope of S.

3.2. Order-k Delaunay diagram and k-set polytope

There exists another relation between order-k Delaunay diagram and k-set polytope: The classical Delaunay diagram (i.e., the order-1 Delaunay diagram) in dimension d is the projection of a $(d + 1)$-dimensional convex polyhedral surface [5, 8]. Let us show that the order-k Delaunay diagram is the projection of some faces of a $(d + 1)$-dimensional k-set polytope.

Let F be the $(d + 1)$-dimensional space spanned by E and by an oriented straight line Δ orthogonal to E. Let $\mathcal{P}$ be the surface of E of equation $x_{d+1} - \sum_{i=1}^{d} x_i^2 = 0$. $\mathcal{P}$ is a paraboloid of revolution with axis Δ and is included in the positive half-space of F bounded by E. The mapping φ that associates to every point x of E with coordinates $(x_1, x_2, ..., x_d)$ the point $\varphi(x)$ of F with coordinates $(x_1, x_2, ..., x_d, x_{d+1} = \sum_{i=1}^{d} x_i^2)$ is an orthogonal lift up transformation from E to $\mathcal{P}$. For any non-vertical hyperplane π of F, a point of F is said to be *below* (resp. *above*)

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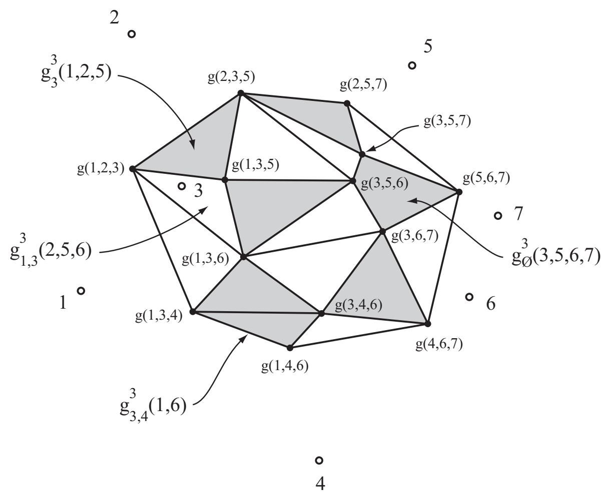

Figure 17. The order-3 Delaunay diagram of the same set of 7 sites as in figure 11. For each region $g^3_P(Q)$, either $|P| = 2$ (white regions), or $|P| \leq 1$ (grey regions). In this latter case, $g^2_P(Q)$ is also an order-2 Delaunay region (see figure 11).

π if it belongs to the open half-space of F bounded by π and that contains the points of Δ at $-\infty$ (resp. $+\infty$).

Given a set S of sites of E, the *lower hull* (resp. *upper hull*) of the k-set polytope $g^k(\varphi(S))$ of $\varphi(S)$ is the set of faces of $g^k(\varphi(S))$ that admit a supporting hyperplane having $g^k(\varphi(S))$ above (resp. below) it.

Theorem 6. *(i) The order-k Delaunay diagram of S is the orthogonal projection on E of the lower hull of the k-set polytope of $\varphi(S)$.*

(ii) The projection of the upper hull of the k-set polytope of $\varphi(S)$ is the image of the order-$(n-k)$ Delaunay diagram of S by the homothety of ratio $-(n-k)/k$ centered at $g(S)$.

Proof. (i) The centroids of the elements of $^k(S)$ are orthogonal projections on E of the centroids of the elements of $^k(\varphi(S))$. Thus, the orthogonal projection on E of the lower hull of $g^k(\varphi(S))$ is a partition of the convex hull of the centroids of the elements of $^k(S)$, that is, a partition of $\overline{g^k(S)}$. Now, by theorem 5, the order-k Delaunay diagram of S is also a partition of $\overline{g^k(S)}$ and, in order to prove the result, we only need to show that each face of this diagram is the projection of a face of the lower hull of $g^k(\varphi(S))$.

Let (P, Q) be a k-section of S, σ a separating sphere of (P, Q) in S, $(c_1, c_2, ..., c_d)$ the coordinates of the center c of σ, and r its radius. For every point x of σ with coordinates $(x_1, x_2, ..., x_d)$, the point $\varphi(x)$ satisfies

$$\sum_{i=1}^{d}(x_i - c_i)^2 - r^2 = 0$$

$$x_{d+1} - \sum_{i=1}^{d} x_i^2 = 0.$$

Consequently, $\varphi(x)$ belongs to the non-vertical hyperplane π of equation

$$x_{d+1} - \sum_{i=1}^{d} 2c_i x_i + \sum_{i=1}^{d} c_i^2 - r^2 = 0.$$

Moreover, $x_{d+1} - \sum_{i=1}^{d} 2c_i x_i + \sum_{i=1}^{d} c_i^2 - r^2$ and $\sum_{i=1}^{d}(x_i - c_i)^2 - r^2$ have the same sign. Thus, for any point x of σ^+ (resp σ^-), $\varphi(x)$ is below (resp. above) π. It follows that $(\varphi(P), \varphi(Q))$ is a generalized k-set of $\varphi(S)$ of separating hyperplane π.

Hence, by lemma 3, $g^k_{\varphi(P)}(\varphi(Q))$ is included in a hyperplane μ parallel to π and, since $\varphi(P)$ is below π, $g^k(\varphi(S))$ is above μ, by the proof of lemma 4. Hence, $g^k_{\varphi(P)}(\varphi(Q))$ is a face of the lower hull of $g^k(\varphi(S))$.

Furthermore, $T \in {}^k(P, Q)$ if and only if $\varphi(T) \in {}^k(\varphi(P), \varphi(Q))$, and $g(T)$ is the orthogonal projection of $g(\varphi(T))$ on E. Thus $g^k_P(Q)$ is the orthogonal projection of $g^k_{\varphi(P)}(\varphi(Q))$ on E (see figure 18).

(ii) In the same way, the projection on E of the upper hull of $g^k(\varphi(S))$ is a partition of $\overline{g^k(S)}$. Since T is an element of $^k(S)$ if and only if $S \setminus T$ is an element of $^{(n-k)}(S)$ and since $g(T)$ is the image of $g(S \setminus T)$ by the homothety $\mathcal{H}$ of ratio $-(n-k)/k$ centered at $g(S)$, the image of the order-$(n-k)$ Delaunay diagram by $\mathcal{H}$ is also a partition of $\overline{g^k(S)}$. Thus we are left to prove that every face of this partition is the projection of a face of the upper hull of $g^k(\varphi(S))$.

As in (i), for every $(n-k)$-section (P, Q) of S, $(\varphi(P), \varphi(Q))$ is a generalized $(n-k)$-set of $\varphi(S)$ and if π is one of its separating hyperplanes, $\varphi(P)$ and $\varphi(S \setminus (P \cup Q))$ are respectively below and above π. Moreover, if Q is empty, $|\varphi(S \setminus (P \cup Q))| = |S \setminus (P \cup Q)| = k$ and otherwise, $|\varphi(S \setminus (P \cup Q))| < k < |\varphi(S \setminus P)|$. Thus, $(\varphi(S \setminus (P \cup Q)), \varphi(Q))$ is a generalized k-set of $\varphi(S)$. By the proof of lemma 4, $g^k(\varphi(S))$ is below the hyperplane parallel to π that contains $g^k_{\varphi(S\setminus(P\cup Q))}(\varphi(Q))$ and $g^k_{\varphi(S\setminus(P\cup Q))}(\varphi(Q))$ is a face of the upper hull of $g^k(\varphi(S))$. Moreover, $T \in {}^{(n-k)}(P, Q)$ if and only if $\varphi(S \setminus T) \in {}^k(\varphi(S \setminus (P \cup Q)), \varphi(Q))$ and the projection on E of $g(\varphi(S \setminus T))$ is the image $g(S \setminus T)$ of $g(T)$ by $\mathcal{H}$. Thus, the projection of $g^k_{\varphi(S\setminus(P\cup Q))}(\varphi(Q))$ is the image of $g^{n-k}_P(Q)$ by $\mathcal{H}$. $\square$

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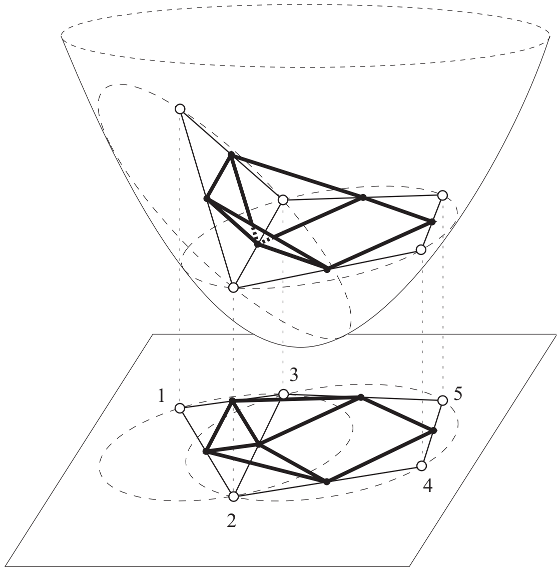

Figure 18. The order-1 Delaunay diagram of a set $S = \{1,2,3,4,5\}$ of planar sites and the lower hull of the 1-set polytope of $\varphi(S)$, in thin lines. The order-2 Delaunay diagram of S and the lower hull of $g^2(\varphi(S))$, in thick lines.

4. Order-k Delaunay diagram and order-k Voronoi diagram

The order-1 Delaunay diagram of S admits a well known dual, the Voronoi diagram of S. The Voronoi diagram is a partition of space E whose every region is the set of points of E strictly closer to a given site of S than to any other. The Voronoi diagram is an orthogonal dual of the Delaunay diagram in the sense that, to every j-dimensional face ($0 < j < d$) of one diagram corresponds an orthogonal $(d-j)$-dimensional face of the other one.

The order-k Voronoi diagram is a generalization of the Voronoi diagram in which every region is the set of points of E having the same k closest neighbors in S. Thus, to construct the order-k Voronoi diagram, one needs to find for every point x in E the subset T of k nearest sites of x. In order that such a set T exists, x has to be the center of a sphere σ that strictly separates T from $S \setminus T$, i.e. σ is a separating sphere of the k-section $(T, \emptyset)$. In the other cases, the k^{th} and $(k+1)^{th}$ nearest sites of x are at the same distance from x and x is the center of a sphere σ that passes through a set Q of at least two sites and has a set P of at most $k-1$ sites inside. More precisely, $|P| < k < |P \cup Q|$ and (P, Q) is a k-section of S of separating sphere σ. This leads to the following definition:

For every k-section (P, Q) of S, the set $f_P^k(Q)$ of centers of all separating spheres of (P, Q) is called an *order-k Voronoi face* of S. Thus, by denoting $d(x, T)$ (resp. $dmax(x, T)$) the minimal (resp. maximal) distance from a point x of E to the sites of a subset T of S, $f_P^k(Q)$ is the set of points of E such that

$$\begin{aligned} f_P^k(Q) = \{\ & x \in E; \\ & dmax(x, P) < d(x, Q) = dmax(x, Q) \\ & \qquad < d(x, S \setminus (P \cup Q))\} \end{aligned}$$

when P, Q and $S \setminus (P \cup Q)$ are non-empty.

If Q is empty, we get the classical definition of the order-k Voronoi region of P:

$$f_P^k(\emptyset) = \{x \in E; dmax(x, P) < d(x, S \setminus P)\}.$$

Since every point in E is the center of a separating sphere of one and only one k-section of S, the set of order-k Voronoi faces forms a partition of E. All that remains to be proven is that the faces $f_P^k(Q)$ with $Q \neq \emptyset$ are really the faces of the order-k Voronoi regions (see figure 19).

For every subset Q of cospherical sites of S, let $bis(Q) = \{x \in E; d(x, Q) = dmax(x, Q)\}$ be the bisector of Q.

Lemma 6. *(i) $f_P^k(\emptyset)$ is an open, connected, and convex region of E,*

(ii) if $0 < dim(Q) < d$, $f_P^k(Q)$ is an open, connected, and convex subset of $bis(Q)$ and $dim(f_P^k(Q)) = d - dim(Q)$,

(iii) if $dim(Q) = d$, $f_P^k(Q)$ is a point of E.

Proof. (i) $f_P^k(\emptyset) = \{x \in E; dmax(x, P) < d(x, S \setminus P)\}$ is the intersection of the open half-spaces $\{x \in E; d(x, p) < d(x, s)\}$ with $p \in P$ and $s \in S \setminus P$. Thus, $f_P^k(\emptyset)$ is an open, connected, and convex d-dimensional subset of E.

(ii) If $0 < dim(Q) < d$ and if neither P nor $S \setminus (P \cup Q)$ are empty,

$$\begin{aligned} f_P^k(Q) = \{\ & x \in E; \\ & dmax(x, P) < d(x, Q) = dmax(x, Q) \\ & \qquad < d(x, S \setminus (P \cup Q))\} \\ = {} & f_P^k(\emptyset) \cap bis(Q) \cap f_{P \cup Q}^k(\emptyset). \end{aligned}$$

Setting $f_\emptyset^k(Q) = f_S^k(\emptyset) = E$, the relation holds even if $P = \emptyset$ and/or $S \setminus (P \cup Q) = \emptyset$.

Hence, in every case, $f_P^k(Q)$ is an open, connected, and convex subset of the bisector of Q of dimension $dim(bis(Q)) = d - dim(Q)$.

(iii) If $dim(Q) = d$, $f_P^k(Q)$ is the center of the unique separating sphere of (P, Q) and is therefore a point of E. $\square$

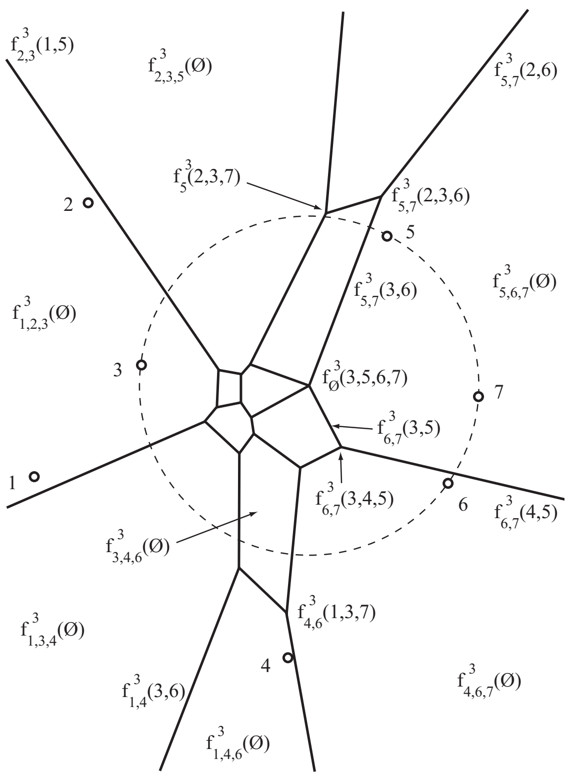

Figure 19. The order-3 Voronoi regions, edges, and vertices of the set of sites of figure 11.

Lemma 7. *The face $f_P^k(Q)$ is unbounded if and only if the k-section (P,Q) is unbounded.*

Proof. Every unbounded k-section (P,Q) admits a separating sphere with unbounded radius. Since its center belongs to $f_P^k(Q)$, $f_P^k(Q)$ is unbounded.

Conversely, if $f_P^k(Q)$ is unbounded, there exists a separating sphere of (P,Q) whose center can tend toward infinity. The radius of such a sphere is then unbounded and so is (P,Q). $\square$

Theorem 7. *For every k-section (P,Q) of S such that $dim(Q) < d$, the faces of $f_P^k(Q)$ are the order-k Voronoi faces $f_{P'}^k(Q')$ such that $(P',Q') \neq (P,Q)$, $P' \subseteq P$, and $P \cup Q \subseteq P' \cup Q'$.*

Proof. (i) If (P',Q') is a k-section of S such that $(P',Q') \neq (P,Q)$, $P' \subseteq P$, and $P \cup Q \subseteq P' \cup Q'$, every separating sphere σ' of (P',Q') passes through all sites of Q and through at least one other site of S and is such that $P \subset \overline{\sigma'^+}$ and $S \setminus (P \cup Q) \subset \overline{\sigma'^-}$. Every point x of $f_{P'}^k(Q')$ being the center of such a sphere, $dmax(x,P) \leq d(x,Q) = dmax(x,Q) \leq d(x, S \setminus (P \cup Q))$ when P, Q, and $S \setminus (P \cup Q)$ are not empty. Then x belongs to $\overline{f_P^k(Q)}$. The result holds even if $P = \emptyset$, $Q = \emptyset$, and/or $S \setminus (P \cup Q) = \emptyset$. Since $f_P^k(Q) \cap f_{P'}^k(Q') = \emptyset$, $f_{P'}^k(Q')$ is included in the boundary of $f_P^k(Q)$.

(ii) Conversely, every point x of the boundary of $f_P^k(Q)$ is the center of a sphere σ that passes through the sites of Q and through at least one other site of S. The sites of P belong to $\overline{\sigma^+}$ and the sites of $S \setminus (P \cup Q)$ belong to $\overline{\sigma^-}$. Setting $Q' = \sigma \cap S$ and $P' = \sigma^+ \cap S$, it follows that $(P',Q') \neq (P,Q)$, $P' \subseteq P$, $P \cup Q \subseteq P' \cup Q'$, and $|P'| < k < |P' \cup Q'|$. Thus (P',Q') is a k-section of S of separating sphere σ whose center is the point x. Hence, x belongs to $f_{P'}^k(Q')$.

(iii) If h is a face of $f_P^k(Q)$, it follows from (i) and (ii) that, when $dim(h) = 0$, h is an order-k Voronoi vertex. When $dim(h) > 0$, h is composed of a set of order-k Voronoi faces. Let us prove by contradiction that this set is reduced to a unique element. If h contains more than one order-k Voronoi face then, since the number of these faces is finite, h contains at least two $dim(h)$-dimensional faces $f_{P_1}^k(Q_1)$ and $f_{P_2}^k(Q_2)$ that are incident in h to a same face $f_{P_3}^k(Q_3)$ of dimension strictly less than $dim(h)$. From (i) and (ii), (P,Q), (P_1,Q_1), (P_2,Q_2), and (P_3,Q_3) are thus pairwise distinct k-sections such that:

– $P_1 \subseteq P$ and $P \cup Q \subseteq P_1 \cup Q_1$,

– $P_2 \subseteq P$ and $P \cup Q \subseteq P_2 \cup Q_2$,

– $P_3 \subseteq P_1$ and $P_1 \cup Q_1 \subseteq P_3 \cup Q_3$,

– $P_3 \subseteq P_2$ and $P_2 \cup Q_2 \subseteq P_3 \cup Q_3$.

By theorem 4, the k-set polytopes $g_{P_1}^k(Q_1)$ and $g_{P_2}^k(Q_2)$ are then two faces of $g_{P_3}^k(Q_3)$ incident to $g_P^k(Q)$. Now, since $f_{P_1}^k(Q_1)$ and $f_{P_2}^k(Q_2)$ are included in a common face h and have same dimension as h, their affine hulls are equal. By lemma 6, it follows that $bis(Q_1) = bis(Q_2)$ and therefore $aff(Q_1)$ is parallel to $aff(Q_2)$. By lemma 3, $g_{P_1}^k(Q_1)$ and $g_{P_2}^k(Q_2)$ are parallel too and, since they are incident to the same k-set polytope $g_P^k(Q)$, their affine hulls are equal. It follows that $g_{P_1}^k(Q_1)$ and $g_{P_2}^k(Q_2)$ are included in a same face of the k-set polytope $g_{P_3}^k(Q_3)$, which is impossible since every face of $g_{P_3}^k(Q_3)$ is a unique k-set polytope, by theorem 4. It follows that h is a unique order-k Voronoi face.

From (ii) this face is of the form $f_{P'}^k(Q')$ with $(P',Q') \neq (P,Q)$, $P' \subseteq P$, and $P \cup Q \subseteq P' \cup Q'$. Moreover, from (i), every face of this form is a face of $f_P^k(Q)$. $\square$

Theorem 8. *The order-k Delaunay diagram is the orthogonal dual of the order-k Voronoi diagram.*

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Proof. For every k-section (P, Q) of S, the mapping ψ that associates $f_P^k(Q)$ to $g_P^k(Q)$ is bijective. By theorems 4 and 7, if (P, Q) and (P', Q') are two k-sections of S such that $f_{P'}^k(Q')$ is a face of $f_P^k(Q)$ then $\psi(f_P^k(Q))$ is a face of $\psi(f_{P'}^k(Q'))$. Hence, the two partitions are dual one from the other.

Moreover, by lemma 6, if $0 < dim(Q) < d$, $f_P^k(Q)$ is orthogonal to $aff(Q)$ and consequently to $g_P^k(Q)$ by lemma 3. Thus, the duality between the two partitions is orthogonal (see figure 20). $\square$

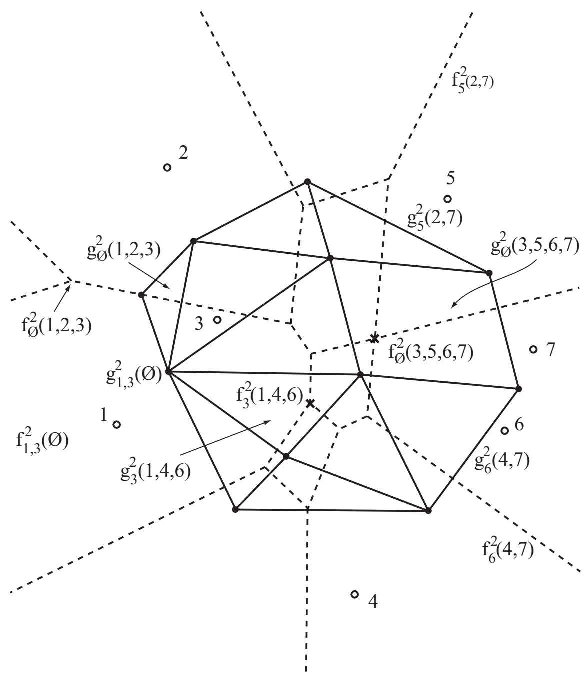

Figure 20. Orthogonal duality between order-2 Delaunay and Voronoi diagrams.

5. Conclusion

In this paper, we have introduced the notions of k-couple of a set of sites and of k-set polytope of such a k-couple. To begin with, we have studied a subset of k-couples, the generalized k-sets, which are defined by separating hyperplanes. We have shown that the k-set polytopes of these k-couples are the faces of the k-set polytope of S. Afterwards, we have considered an other subset of k-couples, the k-sections, which are defined by separating spheres. More particularly, we have shown that the k-set polytopes of these k-sections form the order-k Delaunay diagram, an orthogonal dual of the order-k Voronoi diagram.

The simultaneous studying of these notions allowed us to clarify the close relationship between k-set polytopes and order-k Delaunay diagrams. It also allows to envisage extensions using other kinds of separating surfaces than planes or spheres.

The enumerations of the faces of the k-set polytopes given by Andrzejak and Welzl [3] and our results on order-k Delaunay diagrams, should allow to find new relations between the numbers of faces of order-k Voronoi diagrams and, possibly, help to solve the open problem of the size of these diagrams in higher dimensions.

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