k-Sets of Convex Inclusion Chains of Planar Point Sets

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Abstract. Given a set V of n points in the plane, we introduce a new number of k-sets that is an invariant of V: the number of k-sets of a convex inclusion chain of V. A convex inclusion chain of V is an ordering $(v_1, v_2, ..., v_n)$ of the points of V such that no point of the ordering belongs to the convex hull of its predecessors. The k-sets of such a chain are then the distinct k-sets of all the subsets $\{v_1, ..., v_i\}$, for all i in $\{k+1, ..., n\}$. We show that the number of these k-sets depends only on V and not on the chosen convex inclusion chain. Moreover, this number is surprisingly equal to the number of regions of the order-k Voronoi diagram of V. As an application, we give an efficient on-line algorithm to compute the k-sets of the vertices of a simple polygonal line, no vertex of which belonging to the convex hull of its predecessors on the line.

1 Introduction

Given a finite set V of n points in the Euclidean plane (no three of them being colinear) and an integer k ($0 < k < n$), the k-sets of V are the subsets of k points of V that can be strictly separated from the rest by a straight line. Due to the various applications of k-sets, the problems of constructing and of counting them have been extensively studied in computational and combinatorial geometry. Dey [5] has shown that the number $\gamma^k(V)$ of k-sets of a set V of n points in the plane is at most $O(nk^{\frac{1}{3}})$ and Tóth [12] has shown how to construct point sets with $n2^{\Omega(\sqrt{\log k})}$ k-sets. Narrowing the gap between these two bounds remains an important open problem. More precise results have been obtained by adding up the number of k-sets for different values of k. Peck [11] has shown that the number of $(\leq k)$-sets of V, i.e. the sum of the numbers $\gamma^i(V)$ over all i in $\{1, ..., k\}$, is bounded by kn and that this bound is tight. In this paper we propose a different approach which consists in fixing k and summing the number of k-sets over different subsets of V. To this aim, we define the notion of convex inclusion chain of the point set V which is an ordering $\mathcal{V} = (v_1, v_2, ..., v_n)$ of the points of V such that, for every $i \in \{2, ..., n\}$, v_i does not belong to the convex hull $conv(S_{i-1})$ (with $S_i = \{v_1, ..., v_i\}$, for all $i \in \{1, ..., n\}$). The set of k-sets of the convex inclusion chain $\mathcal{V}$ is then the set of distinct k-sets of $S_{k+1}, S_{k+2}, ..., S_n$.

The main result of this paper is that the number of k-sets of a convex inclusion chain of V is an invariant of the set V, that is, it does not depend on the choice

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of the convex inclusion chain. To prove this result, we use the notion of k-set polygon introduced by Edelsbrunner, Valtr, and Welzl [6]: the k-set polygon $g^k(V)$ of V is the convex hull of the centroids of all the subsets of k elements of V. Andrzejak and Fukuda [1] have shown that the vertices of this k-set polygon are the centroids of the k-sets of V. Thus, counting the number of k-sets comes to count the number of vertices or edges of the k-set polygon. In particular, we show that, given a point v not belonging to $conv(V)$, the edges of $g^k(V)$ that are not edges of $g^k(V \cup \{v\})$ form a connected polygonal line on the boundary $\delta(g^k(V))$ of $g^k(V)$. This generalizes a result which is well known in the case of convex hulls, that is, for $k = 1$. Using this result we show that:

Theorem 1. *Any convex inclusion chain of a planar set V of n points admits $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} \gamma^j(V)$ k-sets (with $\sum_1^0 = 0$).*

Surprisingly, this number is independent of the choice of $\mathcal{V}$ and is also equal to the number of regions of the order-k Voronoi diagram of V (see Lee [7]).

The best worst-case algorithm to construct the k-sets of a set V of n points in the plane has been given by Cole, Sharir, and Yap and runs in $O(n \log n + \gamma^k(V) \log^2 k)$ time [4] (for bigger values of k this can be improved to $O(n \log n + \gamma^k(V) \log^{1+\varepsilon} n)$ [3]).

In the second part of this paper we give an algorithmic method to update the set of k-sets of V when a point v that does not belong to $conv(V)$ is added. We apply this result to the on-line construction of the k-set polygon of a simple polygonal line $\mathcal{V}$ in the particular case where no vertex of $\mathcal{V}$ belongs to the convex hull of its predecessors on $\mathcal{V}$. This algorithm generalizes Melkman's on-line algorithm which constructs the convex hull of a simple polygonal line in linear time [8]. We show that:

Theorem 3. *The k-set polygon of the polygonal line $\mathcal{V}$ can be constructed on-line in $O(k(n-k) \log^2 k)$ time.*

The cost per created k-set is $O(\log^2 k)$, the same as in the algorithm of Cole, Sharir, and Yap [4].

2 Counting k-Sets of Convex Inclusion Chains

Throughout this paper we will consider V to be a finite set of $|V| = n$ points in the Euclidean plane such that $n \geq 2$ and no 3 points of V are colinear. k will be an integer of $\{1, ..., n-1\}$. The aim of this section is to count the number of k-sets of a convex inclusion chain of V. To this end we will use the boundary $\delta(g^k(V))$ of the k-set polygon of V. This boundary is considered to be oriented in counter clockwise direction. Moreover, given two points s and t of V, we denote by st the closed oriented segment with endpoints s and t, by (st) the oriented straight line generated by st, and by $(st)^+$ (resp. $(st)^-$) the open half plane on the left (resp. right) of (st). $\overline{(st)^+}$ and $\overline{(st)^-}$ denote the closure of $(st)^+$ and $(st)^-$.

Let us first recall two important properties of the vertices and edges of k-set polygons given by Andrzejak and Fukuda [1], and by Andrzejak and Welzl [2] (see Fig. 1 for an illustration).

Proposition 1. *The centroid $g(T)$ of a subset T of k points of V is a vertex of $g^k(V)$ if and only if T is a k-set of V. Moreover, the centroids of distinct k-sets are distinct vertices.*

Proposition 2. *T and T' are two k-sets of V such that $g(T)g(T')$ is an oriented edge of $g^k(V)$ if and only if there exist two points s and t of V and a subset P of $k-1$ points of V such that $T = P \cup \{s\}$, $T' = P \cup \{t\}$, and $V \cap (st)^- = P$.*

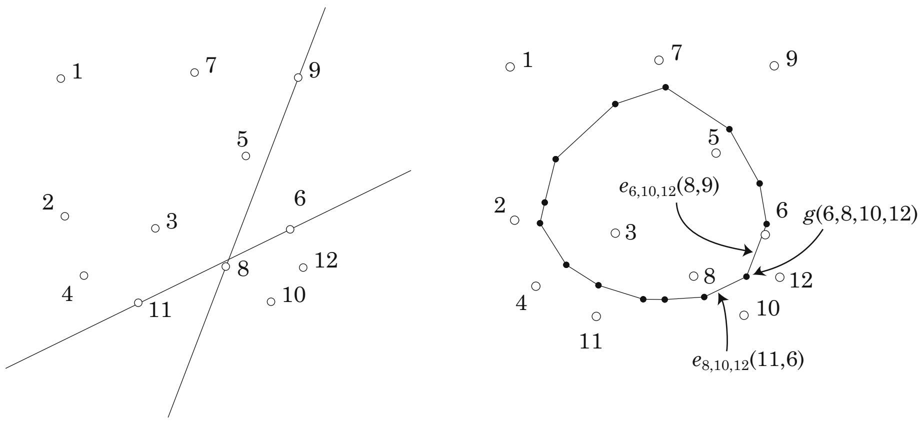

Fig. 1. A set of 12 points and its 4-set polygon

From now on, any oriented edge $g(P \cup \{s\})g(P \cup \{t\})$ of $g^k(V)$ will be denoted by $e_P(s,t)$.

Notice that, in the particular case where $k = 1$, $g^k(V)$ is the convex hull of V and its edges are of the form $e_\emptyset(s,t)$. When V is reduced to two points s and t, $g^1(V)$ admits exactly two oriented edges $e_\emptyset(s,t) = st$ and $e_\emptyset(t,s) = ts$.

We now characterize the edges of the k-set polygon that have to be created and those that have to be removed, when a new point is added (see Fig. 2). The following lemmas can easily be deduced from Proposition 2:

Lemma 1. *If $k < n-1$ then, for any subset S of V such that $k < |S| < n$ and for any point v of $V \setminus S$, an edge $e_P(s,t)$ of $g^k(S)$ is also an edge of $g^k(S \cup \{v\})$ if and only if $v \in (st)^+$.*

Lemma 2. *For any subset S of V such that $k \le |S| < n$ and for any point v of $V \setminus S$,*

(i) an edge $e_P(s,t)$ of $g^k(S \cup \{v\})$ is not an edge of $g^k(S)$ if and only if $v \in P \cup \{s,t\}$,

(ii) if $v \notin conv(S)$, $g^k(S \cup \{v\})$ admits one and only one edge of the form $e_P(s,t)$ with $s = v$ (resp. $t = v$).

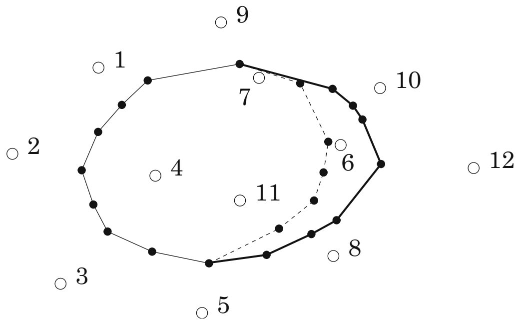

Fig. 2. The 4-set polygon of $S = \{1, ..., 11\}$ and the 4-set polygon of $S \cup \{12\}$. The edges of $g^k(S)$ that are not edges of $g^k(S \cup \{12\})$ are in dashed lines and the edges of $g^k(S \cup \{12\})$ that are not edges of $g^k(S)$ are in bold lines.

Note that, in Lemma 2, the definition of the k-set polygon has implicitly be extended to the case $k = |S|$. In this case, $g^k(S)$ is a unique point of the plane (the centroid of S) and, therefore, it admits no edge. This extended definition will help to simplify the proof of Proposition 4.

Proposition 3. *If $k < n - 1$ then, for any subset S of V such that $k < |S| < n$ and for any point v of $V \setminus conv(S)$,*

(i) the edges of $g^k(S \cup \{v\})$ that are not edges of $g^k(S)$ form an open connected polygonal line with at least two edges, whose first (resp. last) edge in counter clockwise direction is the unique edge of $g^k(S \cup \{v\})$ of the form $e_P(s,t)$ with $t = v$ (resp. $s = v$).

(ii) the edges of $g^k(S)$ that are not edges of $g^k(S \cup \{v\})$ form an open connected and non empty polygonal line.

Proof. (i) From Lemma 2, the set $\mathcal{C}$ of edges of $g^k(S \cup \{v\})$ that are not edges of $g^k(S)$ admits at least two edges. It can also be shown that at least one edge of $g^k(S)$ is an edge of $g^k(S \cup \{v\})$ too. Thus, $\mathcal{C}$ admits at least one edge $e_P(s,t)$ whose first endpoint is a vertex of $g^k(S)$, i.e. $v \notin P \cup \{s\}$. Hence, from Lemma 2, $t = v$ and $e_P(s,t)$ is the only edge of $g^k(S \cup \{v\})$ of the form $e_P(s,v)$. In the same way, there is a unique edge of $\mathcal{C}$ whose second endpoint is a vertex of $g^k(S)$ and this edge is of the form $e_P(v,t)$. It follows that $\mathcal{C}$ is an open connected polygonal line whose first (resp. last) edge in counter clockwise direction is of the form $e_P(s,v)$ (resp. $e_P(v,t)$).

(ii) Straightforward from (i). □

This proposition generalizes a result which is well known in the case $k = 1$: The edges of a convex hull that are visible from a point outside of the hull form an open connected and non empty polygonal line. Moreover, if we want to update the convex hull after the insertion of such a point, two new edges have to be created. This means that the incremental construction of the convex hull of n points, in such a way that every newly inserted point does not belong to the current convex hull, constructs always $2(n-1)$ edges (two per inserted point except for the first one). We now generalize this last result for $k \neq 1$.

Let $(v_1, v_2, ..., v_n)$ be a convex inclusion chain of V, that is, an ordering of the points of V such that, for every $i \in \{1, ..., n-1\}$, $v_{i+1} \notin conv(S_i)$ (with $S_i = \{v_1, ..., v_i\}$, for every $i \in \{1, ..., n\}$).

For every $k \in \{1, ..., n-1\}$ and for every $i \in \{k+1, ..., n\}$, let c_i^k denote the number of edges of $g^k(S_i)$ that are not edges of $g^k(S_{i-1})$, i.e. the number of edges to create while constructing the k-set polygon of $S_i = S_{i-1} \cup v_i$ from the k-set polygon of S_{i-1}. Since the number of edges of $g^k(S_k)$ is zero, $c^k = \sum_{i=k+1}^{n} c_i^k$ is the total number of edges to be created by an algorithm that incrementally constructs $g^k(V)$ by successively determining $g^k(S_k)$, $g^k(S_{k+1})$, ..., $g^k(S_n)$.

From Proposition 1, for every $j \in \{1, ..., n-1\}$, the number of edges (i.e. the number of vertices) of the j-set-polygon of V is equal to the number $\gamma^j(V)$ of j-sets of V.

Proposition 4. *$c^1 = 2(n-1)$ and $c^k = k(2n-k-1) - \sum_{j=1}^{k-1} \gamma^j(V)$ if $1 < k < n$.*

Proof. From Lemma 2, for every $i \in \{k+1, ..., n\}$, $g^k(S_{i-1} \cup \{v_i\})$ admits at least two edges that are not edges of $g^k(S_{i-1})$. These two edges are of the form $e_Q(v_i, t)$ and $e_P(s, v_i)$. All other edges of $g^k(S_{i-1} \cup \{v_i\})$ that are not edges of $g^k(S_{i-1})$ are of the form $e_{P'}(s', t')$ with $v_i \in P'$. If $k = 1$, no such other edge exists since $P = \emptyset$. Thus $c_i^1 = 2$, for every $i \in \{2, ..., n\}$, and

$$c^1 = \sum_{i=2}^{n} 2 = 2(n-1) \ .$$

If $k \in \{2, ..., n-1\}$, from Lemma 1, $e_{P'}(s', t')$ is an edge of $g^k(S_{i-1} \cup \{v_i\})$ with $v_i \in P'$ if and only if $e_{P' \setminus \{v_i\}}(s', t')$ is an edge of $g^{k-1}(S_{i-1})$ and is not an edge of $g^{k-1}(S_{i-1} \cup \{v_i\})$. Thus, denoting by d_i^{k-1} the number of edges of $g^{k-1}(S_{i-1})$ that are not edges of $g^{k-1}(S_i)$, we have $c_i^k = 2 + d_i^{k-1}$. It follows that

$$c^k = \sum_{i=k+1}^{n} c_i^k = 2(n-k) + \sum_{i=k+1}^{n} d_i^{k-1} \ .$$

Now, since the number of edges of $g^{k-1}(S_{k-1})$ is zero, we have $d_k^{k-1} = 0$ and $\sum_{i=k+1}^{n} d_i^{k-1}$ is the total number of edges to be deleted by an algorithm that incrementally constructs $g^{k-1}(V)$ by successively determining $g^{k-1}(S_{k-1})$, $g^{k-1}(S_k)$, ..., $g^{k-1}(S_n)$. Thus

$$\sum_{i=k+1}^{n} d_i^{k-1} = c^{k-1} - \gamma^{k-1}(V)$$

and

$$c^k = 2(n-k) + c^{k-1} - \gamma^{k-1}(V) \ .$$

Solving this induction relation we get

$$c^k = (k-1)(2n-k-2) + c^1 - \sum_{j=1}^{k-1} \gamma^j(V) = k(2n-k-1) - \sum_{j=1}^{k-1} \gamma^j(V) \ .$$

□

The result of this proposition is somewhat surprising since it shows that the number of edges that have to be created for the incremental construction of a k-set polygon does not depend on the order in which the points are treated, provided that every new inserted point does not belong to the convex hull of the previously inserted ones. In addition, since $\sum_{j=1}^{k-1} \gamma^j(V)$ is the number of $(\leq (k-1))$-sets of V and since this number is known to be bounded by $(k-1)n$ (see [11]), it follows that:

Corollary 1. *Any algorithm that incrementally constructs the k-set polygon of n points, so that no point belongs to the convex hull of the points inserted before him, has to create $\Theta(k(n-k))$ edges.*

Now, it is easy to find the number of k-sets of a convex inclusion chain of V:

Theorem 1. *Any convex inclusion chain of a planar set V of n points admits $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} \gamma^j(V)$ k-sets (with $\sum_1^0 = 0$).*

Proof. Taking the previous notations, if $\mathcal{V} = (v_1, v_2, ..., v_n)$ is a convex inclusion chain of V, the number of k-sets of $\mathcal{V}$ is equal to the number of distinct k-set polygon vertices created by an incremental algorithm that successively constructs $g^k(S_{k+1})$, ..., $g^k(S_n)$. The number of vertices of $g^k(S_{k+1})$ is equal to the number c_{k+1}^k of its edges. Moreover, from Proposition 3, for every $i \in \{k+2, ..., n\}$, the edges of $g^k(S_i)$ that are not edges of $g^k(S_{i-1})$ form an open connected and non empty polygonal line. Thus, the number of vertices of this line that are not vertices of $g^k(S_{i-1})$ is $c_i^k - 1$, where c_i^k is the number of edges of the line. It follows that the number of k-sets of $\mathcal{V}$ is $c_{k+1}^k + \sum_{i=k+2}^n (c_i^k - 1)$, that is, from Proposition 4, $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} \gamma^j(V)$. □

According to this theorem, the number of k-sets of a convex inclusion chain of V only depends on the set V and not on the chosen chain. An even more intriguing consequence of the theorem arises from its connection with order-k Voronoi diagrams. The order-k Voronoi diagram of V is a partition of the plane in regions which are the set of points in the plane having the same k nearest neigbours in V. Lee [7] has shown that, if no four points of V are cocircular, the order-k Voronoi diagram of V admits $2kn - n - k^2 + 1 - \sum_{j=1}^{k-1} \gamma^j(V)$ regions; the same number as the one found in Theorem 1. Since a subset of k points of V generates an order-k Voronoi region if and only if it can be separated from the remaining points by a circle, it follows that:

Corollary 2. *Given a set V of points in the plane, no three of them being colinear and no four of them being cocircular, the number of k-sets of a convex inclusion chain of V is equal to the number of subsets of k points of V that can be separated from the remaining by a circle.*

3 Constructing k-Sets of Convex Inclusion Chains

In this section we will consider the construction of the k-sets of a convex inclusion chain of V. In particular, we will show how to update the set of k-sets of a subset S of V when a new point, that does not belong to $conv(S)$, is added. As in the previous section, we will use the k-set polygon of V as a powerful tool.

For $k \in \{1, \dots, |V| - 2\}$, for S a subset of V such that $k < |S| < n$, and for v a point of $V \setminus conv(S)$, we set the following notations:

(i) Let $\mathcal{C}_{S,v}$ (resp. $\mathcal{D}_{S,v}$) denote the counter clockwise oriented polygonal line of edges of $g^k(S \cup \{v\})$ (resp. $g^k(S)$) that are not edges of $g^k(S)$ (resp. $g^k(S \cup \{v\})$).
(ii) Let $T_1, T_2, ..., T_m$ denote the k-sets of S such that $(g(T_1), g(T_2), ..., g(T_m))$ is the ordered sequence of vertices of $\mathcal{D}_{S,v}$ (including its two endpoints).
(iii) For every $i \in \{1, ..., m\}$, let $e_{P_i}(s_i, t_i)$ denote the oriented edge of $g^k(S)$ whose second endpoint is $g(T_i)$ and let $e_{P_{m+1}}(s_{m+1}, t_{m+1})$ denote the oriented edge whose first endpoint is $g(T_m)$.
(iv) Set $\alpha_1 = \omega_m = v$ and, for every $i \in \{2, ..., m\}$, set $\alpha_i = t_i$ and $\omega_{i-1} = s_i$ (see Fig. 3 for an illustration).
(v) For every $i \in \{1, ..., m\}$, if $\alpha_i \neq \omega_i$, $\varphi(T_i)$ denotes the oriented polygonal line that connects α_i to ω_i in counter clockwise direction on $\delta(conv(T_i \cup \{v\}))$ and, if $\alpha_i = \omega_i$, set $\varphi(T_i) = \alpha_i$.
(vi) For every $i \in \{1, ..., m\}$, let $\mathcal{H}_i$ denote the homothety of center $g(T_i \cup \{v\})$ and ratio $-\frac{1}{k}$, that is, for any point x in the plane, $\mathcal{H}_i(x) = g((T_i \cup \{v\}) \setminus \{x\})$.

Notation (v) makes sense since, for every $i \in \{1, ..., m\}$, α_i and ω_i are vertices of $conv(T_i \cup \{v\})$. Indeed, $\alpha_1 = \omega_m = v$ can be separated from S by a straight line and is thus a vertex of $conv(T_1 \cup \{v\})$ and of $conv(T_m \cup \{v\})$. Moreover, from Proposition 2, for every $i \in \{2, ..., m\}$, $T_i \setminus t_i = P_i \subset (s_i t_i)^-$ and, from Lemma 1, $v \in (s_i t_i)^-$. Thus, $\alpha_i = t_i$ is a vertex of $conv(T_i \cup \{v\})$, for all $i \in \{2, ..., m\}$. In the same way, $\omega_i = s_{i+1}$ is a vertex of $conv(T_i \cup \{v\})$, for all $i \in \{1, ..., m-1\}$.

We now show that every vertex $g(T_i)$ of $\mathcal{D}_{S,v}$ can be associated to a subset of $\mathcal{C}_{S,v}$.

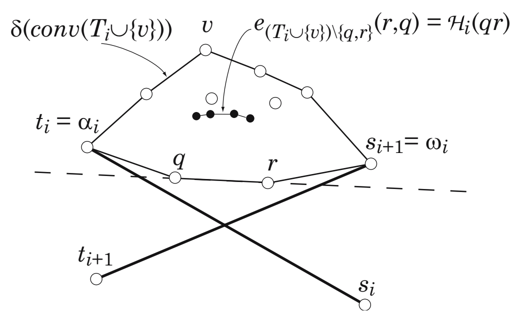

Fig. 3. $s_i t_i$ and $s_{i+1} t_{i+1}$ are such that $e_{P_i}(s_i, t_i)$ and $e_{P_{i+1}}(s_{i+1}, t_{i+1})$ are the two consecutive edges of $\mathcal{D}_{S,v}$ sharing the vertex $g(T_i)$

Lemma 3. *For every $i \in \{1, ..., m\}$ such that $\varphi(T_i)$ is not reduced to a single point and for every oriented edge qr of $\varphi(T_i)$, $\mathcal{H}_i(qr)$ is the edge $e_{T_i \cup \{v\} \setminus \{q,r\}}(r, q)$ of $\mathcal{C}_{S,v}$.*

Proof. (i) We first show that, if $i \in \{2, ..., m-1\}$, then $S \setminus T_i \subset (qr)^-$. Since $g(T_i)$ is the vertex shared by the edges $e_{P_i}(s_i, t_i)$ and $e_{P_{i+1}}(s_{i+1}, t_{i+1})$, we have $g(T_i) = g(P_i \cup \{t_i\}) = g(P_{i+1} \cup \{s_{i+1}\})$ and, from Proposition 1, $T_i = P_i \cup \{t_i\} = P_{i+1} \cup \{s_{i+1}\}$. From Proposition 2, it follows that $T_i \subset \overline{(s_i t_i)^-} \cap \overline{(s_{i+1} t_{i+1})^-}$ and that $S \setminus T_i \subset \overline{(s_i t_i)^+} \cap \overline{(s_{i+1} t_{i+1})^+}$. Thus, $(s_i t_i)$ and $(s_{i+1} t_{i+1})$ are the common tangents of *conv*(T_i) and of *conv*$(S \setminus T_i)$ such that *conv*(T_i) and *conv*$(S \setminus T_i)$ are on both sides of these tangents (see Fig. 3). Moreover, since $\{t_i, s_{i+1}\} \subset T_i$ and $\{s_i, t_{i+1}\} \subset S \setminus T_i$, the intersection point of the segments $s_i t_i$ and $s_{i+1} t_{i+1}$ is either the point $t_i = s_{i+1}$ or a point of $(s_{i+1} t_i)^+$. Since $\varphi(T_i)$ is not reduced to a single point, $t_i \neq s_{i+1}$ and, since $v \notin$ *conv*(S), it follows from Lemma 1 that $v \in (s_i t_i)^- \cap (s_{i+1} t_{i+1})^- \cap (s_{i+1} t_i)^-$. Thus, $(s_i t_i)$ and $(s_{i+1} t_{i+1})$ are also common tangents of *conv*$(T_i \cup \{v\})$ and of *conv*$(S \setminus T_i)$. The edges of $\varphi(T_i)$ are then the edges of *conv*(T_i) included in $(s_{i+1} t_i)^+$ and the slopes of the oriented straight lines generated by such edges are comprised between the slopes of $(t_i s_i)$ and $(t_{i+1} s_{i+1})$. Thus, the edges of $\varphi(T_i)$ are the edges of *conv*$(T_i \cup \{v\})$ visible from every point of $\overline{(s_i t_i)^+} \cap \overline{(s_{i+1} t_{i+1})^+}$ and, since $S \setminus T_i \subset \overline{(s_i t_i)^+} \cap \overline{(s_{i+1} t_{i+1})^+}$, it follows that any edge qr of $\varphi(T_i)$ is such that $S \setminus T_i \subset (qr)^-$.

In a similar way, we can show that $S \setminus T_1 \subset (qr)^-$ and $S \setminus T_m \subset (qr)^-$.

(ii) Since every edge qr of $\varphi(T_i)$ is an edge of *conv*$(T_i \cup \{v\})$, for every $i \in \{1, ..., m\}$, we have $(T_i \cup \{v\}) \setminus \{q, r\} \subset (qr)^+$. Moreover, since $|(T_i \cup \{v\}) \setminus \{q, r\}| = k-1$, it follows from (i) and from Proposition 2 that $e_{(T_i \cup \{v\}) \setminus \{q,r\}}(r, q)$ is an edge of $g^k(S \cup \{v\})$ and, from Lemma 2, that this edge belongs to $\mathcal{C}_{S,v}$. Moreover, the endpoints $g((T_i \cup \{v\}) \setminus \{q\})$ and $g((T_i \cup \{v\}) \setminus \{r\})$ of this edge are the respective images of q and r by the homothety $\mathcal{H}_i$ of center $g(T_i \cup \{v\})$ and ratio $-\frac{1}{k}$. Thus $e_{(T_i \cup \{v\}) \setminus \{q,r\}}(r, q) = \mathcal{H}_i(qr)$. □

And thus, the complete characterization of the line $\mathcal{C}_{S,v}$:

Theorem 2. *$\mathcal{C}_{S,v}$ is the sequence of polygonal lines $(\mathcal{H}_1(\varphi(T_1)), ..., \mathcal{H}_m(\varphi(T_m)))$.*

Proof. (i) From Lemma 3, for every $i \in \{1, ..., m\}$, if $\varphi(T_i)$ admits at least one edge, $\mathcal{H}_i(\varphi(T_i))$ is a connected polygonal line included in $\mathcal{C}_{S,v}$. Moreover, for every $j \in \{1, ..., m\}$ such that $j \neq i$, we have $T_i \neq T_j$ and thus, for every edge $q_i r_i$ of $\varphi(T_i)$ and for every edge $q_j r_j$ of $\varphi(T_j)$, $\mathcal{H}_i(q_i r_i) = e_{(T_i \cup \{v\}) \setminus \{q_i, r_i\}}(r_i, q_i)$ and $\mathcal{H}_j(q_j r_j) = e_{(T_j \cup \{v\}) \setminus \{q_j, r_j\}}(r_j, q_j)$ are distinct edges of $\mathcal{C}_{S,v}$. Hence, $\mathcal{H}_i(\varphi(T_i))$ and $\mathcal{H}_j(\varphi(T_j))$ share no edge.

(ii) Let us now show that all the polygonal lines $\mathcal{H}_i(\varphi(T_i))$, $i \in \{1, ..., m\}$, fill $\mathcal{C}_{S,v}$. By definition, the first edge of $\varphi(T_1)$ connects v to a point r of T_1. This edge always exists since the two endpoints $\alpha_1 = v$ and $\omega_1 = s_2$ of $\varphi(T_1)$ are distinct. From Lemma 3 and Proposition 3, $\mathcal{H}_1(vr) = e_{T_1 \setminus \{r\}}(r, v)$ is then the first edge of $\mathcal{C}_{S,v}$ and $\mathcal{H}_1(\varphi(T_1))$ is an initial subsequence of $\mathcal{C}_{S,v}$. In the same way, $\mathcal{H}_m(\varphi(T_m))$ is a final subsequence of $\mathcal{C}_{S,v}$. Moreover, for all $i \in \{1, ..., m-1\}$, $\omega_i = s_{i+1}$ and $\alpha_{i+1} = t_{i+1}$, that is $\mathcal{H}_i(\omega_i) = g((T_i \cup \{v\}) \setminus \{s_{i+1}\})$ and $\mathcal{H}_{i+1}(\alpha_{i+1}) = g((T_{i+1} \cup$

$\{v\}) \setminus \{t_{i+1}\})$. From Proposition 2, $T_i \setminus \{s_{i+1}\} = T_{i+1} \setminus \{t_{i+1}\}$ and it follows that $\mathcal{H}_i(\omega_i) = \mathcal{H}_{i+1}(\alpha_{i+1})$. Finally, $\mathcal{C}_{S,v} = (\mathcal{H}_1(\varphi(T_1)), ..., \mathcal{H}_m(\varphi(T_m)))$. □

The result of this theorem can now be used to develop an algorithm that updates the k-set polygon of S when v is added. Let us first describe the data structure to implement. The boundary of the k-set polygon of S can be stored in a circular list L whose elements represent the edges of $g^k(S)$. To any element e of L which represents an edge $e_P(s,t)$ of $g^k(S)$ are associated the two elements of L that represent the predecessor and the successor of $e_P(s,t)$ on $\delta(g^k(S))$, as well as the two points s and t of S. Note that, from Proposition 2, the k-sets defining two consecutive vertices of $g^k(S)$ differ from each other by one site and thus it suffices to know one k-set T of S and one edge with endpoint $g(T)$ to be able to generate the whole k-sets of S while traversing L. It follows that a k-set polygon with c edges can be stored in a data structure of size $O(c+k)$ and thus provides a compact way to encode the k-sets of a given point set.

In our algorithm we also use a data structure CH that allows dynamic convex hull maintenance. Using results given by Overmars and van Leeuwen [10] (see also Overmars [9]), this structure needs $O(h)$ size to store the convex hull of h points of the plane, allows to get the predeccessor and the successor of any edge in constant time, and can be updated in $O(\log^2 h)$ time after inserting or deleting a point.

For any polygonal line $\mathcal{P}$, let now $|\mathcal{P}|$ denote the number of vertices of $\mathcal{P}$.

Proposition 5. *The edges of $\mathcal{D}_{S,v}$ can be removed from L and the edges of $\mathcal{C}_{S,v}$ inserted in L in $O(|\mathcal{D}_{S,v}| \log^2 k + |\mathcal{C}_{S,v}|)$ time provided that one edge e of L belonging to $\mathcal{D}_{S,v}$ is given, and that the convex hull of one k-set T of S whose centroid is an endpoint of e is stored in CH.*

Proof. From Theorem 2, determining $\mathcal{C}_{S,v}$ comes, for every $i \in \{1, ..., m\}$, to determine $\mathcal{H}_i(\varphi(T_i))$ where $\varphi(T_i)$ is a connected subset of $conv(T_i \cup \{v\})$. Suppose that an edge e of L belonging to $\mathcal{D}_{S,v}$ is given, and that the convex hull of a k-set T of S whose centroid is an endpoint of e is stored in CH. We first show that $conv(T_1 \cup \{v\})$ can be determined from $conv(T)$ in $O(|\mathcal{D}_{S,v}| \log^2 k)$ time. From Lemma 1 and Proposition 3, $\mathcal{D}_{S,v}$ is the polygonal line formed by the edges $e_P(s,t)$ of $g^k(S)$ such that v is on the right of (st). Since the points s and t are associated to the edge $e_P(s,t)$ in L, since constant time is needed to test on which side of (st) v lies, and since the neighbours of any edge in L can also be obtained in constant time, it follows that $e_{P_1}(s_1,t_1)$ can be found, starting from e, in $O(|\mathcal{D}_{S,v}|)$ time. Moreover, from Proposition 2, $T_{i-1} = (T_i \setminus \{t_i\}) \cup \{s_i\}$, for every $i \in \{2, ..., m\}$. Hence $conv(T_{i-1})$ can be computed from $conv(T_i)$ in $O(\log^2 k)$ time. Thus, while searching $e_{P_1}(s_1,t_1)$, $conv(T)$ can be replaced by $conv(T_1)$ in CH in $O(|\mathcal{D}_{S,v}| \log^2 k)$ time and $conv(T_1 \cup \{v\})$ can then be deduced in $O(\log^2 k)$ time. Now, the polygonal line $\varphi(T_1)$ which connects $\alpha_1 = v$ and $\omega_1 = s_2$ on $\delta(conv(T_1 \cup \{v\}))$ can be reported in $O(|\varphi(T_1)|)$ time. From Lemma 3, for every edge qr of $\varphi(T_1)$, the edge $\mathcal{H}_1(qr) = e_{(T_1 \cup \{v\}) \setminus \{q,r\}}(r,q)$ is an edge of $\mathcal{C}_{S,v}$. This comes to insert a new edge in L to which are associated its two neigbours in L as well as the points r and q. Since the edges of $\mathcal{H}_1(\varphi(T_1))$

appear in L in the same order as their corresponding edges on $\varphi(T_1)$, it follows that $\mathcal{H}_1(\varphi(T_1))$ can be inserted in L in $O(|\varphi(T_1)|)$ time. In the same way, for every $i \in \{2, ..., m\}$, $T_i = (T_{i-1} \setminus \{s_i\}) \cup \{t_i\}$ and thus $conv(T_i \cup \{v\})$ can be computed from $conv(T_{i-1} \cup \{v\}))$ in $O(\log^2 k)$ time. $\varphi(T_i)$, which connects $\alpha_i = t_i$ and ω_i, can then be reported in $O(|\varphi(T_i)|)$ time and $\mathcal{H}_i(\varphi(T_i))$ can also be inserted in L in $O(|\varphi(T_i)|)$ time (note that, from Theorem 2, $\mathcal{H}_i(\varphi(T_i))$ follows $\mathcal{H}_{i-1}(\varphi(T_{i-1}))$ in L). Finally, L can be updated after the insertion of v in total $O(|\mathcal{D}_{S,v}| \log^2 k + \sum_{i=1}^{m} |\varphi(T_i)|)$, that is, $O(|\mathcal{D}_{S,v}| \log^2 k + |\mathcal{C}_{S,v}|)$ time. □

Remark 1. Notice that at the end of the algorithm described by Proposition 5, the data structure CH contains the convex hull of T_m, with $g(T_m)$ a common endpoint of $\mathcal{D}_{S,v}$ and $\mathcal{C}_{S,v}$. Moreover, the edge of $\mathcal{C}_{S,v}$ with endpoint $g(T_m)$ is the last edge inserted in L and therefore it can be easily maintained.

We will now show how Proposition 5 can be applied to the on-line construction of the k-set polygon of a planar simple polygonal line $\mathcal{V} = (v_1, v_2, ..., v_n)$ of n vertices which is such that, for every $i \in \{2, ..., n\}$, $v_i \notin conv(S_{i-1})$ (with $S_i = \{v_1, ..., v_i\}$ for every $i \in \{1, ..., n\}$).

Theorem 3. *The k-set polygon of the polygonal line $\mathcal{V}$ can be constructed on-line in $O(k(n-k)\log^2 k)$ time.*

Proof. (i) We first show that the k-set polygon of $S_{k+1} = \{v_1, ..., v_{k+1}\}$ can be computed in $O(k)$ time. From Proposition 2, every edge of $g^k(S_{k+1})$ is of the form $e_{S_{k+1}\setminus\{s,t\}}(s,t)$ where ts is an edge of $conv(S_{k+1})$. Conversely, if ts is an edge of $conv(S_{k+1})$, then $e_{S_{k+1}\setminus\{s,t\}}(s,t)$ is an edge of $g^k(S_{k+1})$. In addition, if $e_{S_{k+1}\setminus\{s',t'\}}(s',t')$ is the successor of $e_{S_{k+1}\setminus\{s,t\}}(s,t)$ on $\delta(g^k(S_{k+1}))$, then $(S_{k+1} \setminus \{s,t\}) \cup \{t\} = (S_{k+1} \setminus \{s',t'\}) \cup \{s'\}$, that is, $s = t'$. ts and $t's'$ are therefore two consecutive edges of $conv(S_{k+1})$ and it follows that constructing $g^k(S_{k+1})$ comes to construct $conv(S_{k+1})$. The convex hull of a simple polygonal line of $k+1$ vertices can be constructed on-line in $O(k)$ time using Melkman's algorithm [8].

Moreover, let $e = e_{S_{k+1}\setminus\{s,t\}}(s,t)$ be an edge of $g^k(S_{k+1})$. Setting $T = S_{k+1} \setminus \{s\}$, $g(T)$ is an endpoint of e and $conv(T)$ can be stored in the data structure CH in $O(k \log^2 k)$ time.

(ii) We now show that, for every $i \in \{k+2, ..., n\}$, $g^k(S_i)$ can be computed from $g^k(S_{i-1})$ in $O((|\mathcal{D}_{S_{i-1},v_i}| + |\mathcal{C}_{S_{i-2},v_{i-1}}|)\log^2 k + |\mathcal{C}_{S_{i-1},v_i}|)$ time (here $\mathcal{C}_{S_k,v_{k+1}}$ denotes the boundary of $g^k(S_{k+1})$).

(ii.1) We first prove that at least one edge of $\mathcal{D}_{S_{i-1},v_i}$ is also an edge of $\mathcal{C}_{S_{i-2},v_{i-1}}$. From the definition of $\mathcal{V}$, v_{i-1} is a vertex of $conv(S_{i-1})$ visible from v_i. Thus, there exists an oriented straight line Δ passing through v_{i-1}, that is not parallel to any straight line passing through any two points of S_{i-1}, and such that $conv(S_{i-1}) \subset \overline{\Delta^+}$ and $v_i \in \Delta^-$. Let Δ' be a straight line parallel to Δ, oriented in the same direction as Δ and such that $|\Delta'^- \cap S_{i-1}| = k$. Let $U = \Delta'^- \cap S_{i-1}$. Let (st) and $(s't')$ be the oriented straight lines tangent to both $conv(U)$ and $conv(S_{i-1} \setminus U)$ such that $\{s', t\} \subseteq U$, $conv(U) \subset \overline{(st)^-}$, and $conv(S_{i-1} \setminus U) \subset \overline{(st)^+}$ (resp. $conv(U) \subset \overline{(s't')^-}$, and $conv(S_{i-1} \setminus U) \subset \overline{(s't')^+}$). Thus, from Proposition 2 and Lemma 2, $e_{U\setminus\{t\}}(s,t)$ and $e_{U\setminus\{s'\}}(s',t')$ are edges of $g^k(S_{i-1})$ that belong to $\mathcal{C}_{S_{i-2},v_{i-1}}$, since $v_{i-1} \in U$. Now, v_i cannot belong to both $(st)^+$ and $(s't')^+$ and, from Lemma 1, at least one of $e_{U\setminus\{t\}}(s,t)$ and $e_{U\setminus\{s'\}}(s',t')$ belongs to $\mathcal{D}_{S_{i-1},v_i}$. Hence, at least one edge of $\mathcal{D}_{S_{i-1},v_i}$ is also an edge of $\mathcal{C}_{S_{i-2},v_{i-1}}$.

(ii.2) Now, from (i) and from Remark 1, after the construction of $g^k(S_{i-1})$, an edge e of $\mathcal{C}_{S_{i-2},v_{i-1}}$ is given and the convex hull of a k-set T whose centroid is a vertex of e is known. From (ii.1), an edge e' of $\mathcal{D}_{S_{i-1},v_i}$ can then be found, starting from e, in $O(|\mathcal{C}_{S_{i-2},v_{i-1}}|)$ time, as in the proof of Proposition 5. The same, the convex hull of a k-set T' whose centroid is a vertex of e' can be constructed, starting from $conv(T)$, in $O(|\mathcal{C}_{S_{i-2},v_{i-1}}|\log^2 k)$ time. Thus, from Proposition 5, for every $i \in \{k+2,...,n\}$, $g^k(S_i)$ can be constructed from $g^k(S_{i-1})$ in $O((|\mathcal{D}_{S_{i-1},v_i}| + |\mathcal{C}_{S_{i-2},v_{i-1}}|)\log^2 k + |\mathcal{C}_{S_{i-1},v_i}|)$ time.

(iii) It follows from (i) and (ii) that the k-set polygon of $\mathcal{V}$ can be constructed on-line in $O(k\log^2 k + \sum_{i=k+2}^{n}((|\mathcal{D}_{S_{i-1},v_i}| + |\mathcal{C}_{S_{i-2},v_{i-1}}|)\log^2 k + |\mathcal{C}_{S_{i-1},v_i}|))$ time. By setting, as in section 1, $c^k = \sum_{i=k+1}^{n} |\mathcal{C}_{S_{i-1},v_i}|$, we have $\sum_{i=k+2}^{n}((|\mathcal{D}_{S_{i-1},v_i}| + |\mathcal{C}_{S_{i-2},v_{i-1}}|)\log^2 k + |\mathcal{C}_{S_{i-1},v_i}|) \leq 2c^k\log^2 k + c^k$. From Proposition 4, c^k is in $O(k(n-k))$ and the time complexity of the algorithm is $O(k(n-k)\log^2 k)$. □

From Corollary 1, any algorithm that incrementally constructs the k-set polygon of the polygonal line $\mathcal{V}$, has to generate $\Omega(k(n-k))$ edges. It follows that the time complexity of the above algorithm, per edge that has to be created, is $O(\log^2 k)$. This complexity can be compared to the one in the algorithm given by Cole, Sharir, and Yap [4] which constructs the set of k-sets of n points in the plane in $O(n\log n + c\log^2 k)$ time, where c is the total number of k-sets of the n points.

4 Conclusion

In this paper we have shown that all the convex inclusion chains of a given set V of points in the plane admit the same number of k-sets. This number is also equal to the number of regions of the order-k Voronoi diagram of V. Up to now we do not know any direct proof of this last result. Such a proof would provide a completely different way to count the number of regions of the order-k Voronoi diagrams in the plane. Studying these relations in higher dimensions would then be of great interest since the size of the order-k Voronoi diagrams is not known in dimension greater than two.

By using the properties of the k-set polygons, we have also given an algorithm to update the set of k-sets of V when a new point that does not belong to $conv(V)$ is added. This algorithm has then be applied to the on-line construction of the k-sets of certain simple polygonal lines. The time complexity of both algorithms is $O(\log^2 k)$ per created edge. This factor comes from the use of the dynamic convex hull data structure of Overmars and van Leeuwen [10]. Chan [3] has given a data structure that allows dynamic maintenance of the convex hull of n points in $O(lg^{1+\varepsilon}n)$ amortized time. Using this data structure, the overall complexity of our second algorithm becomes $O(k(n-k)\lg^{1+\varepsilon} n)$, which is interesting for bigger values of k.

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