Stability of Curvature Measures

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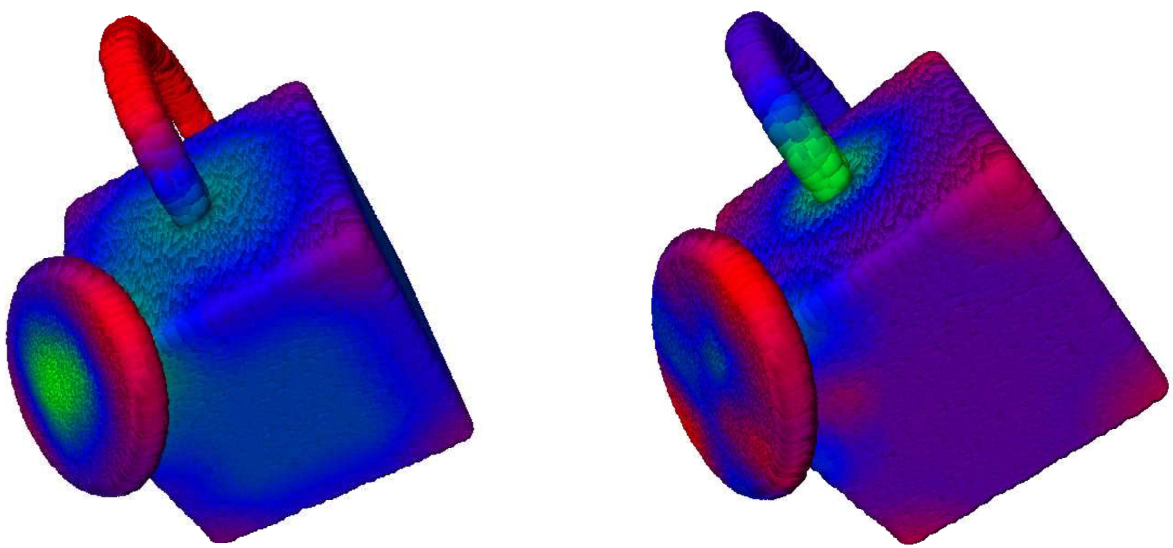
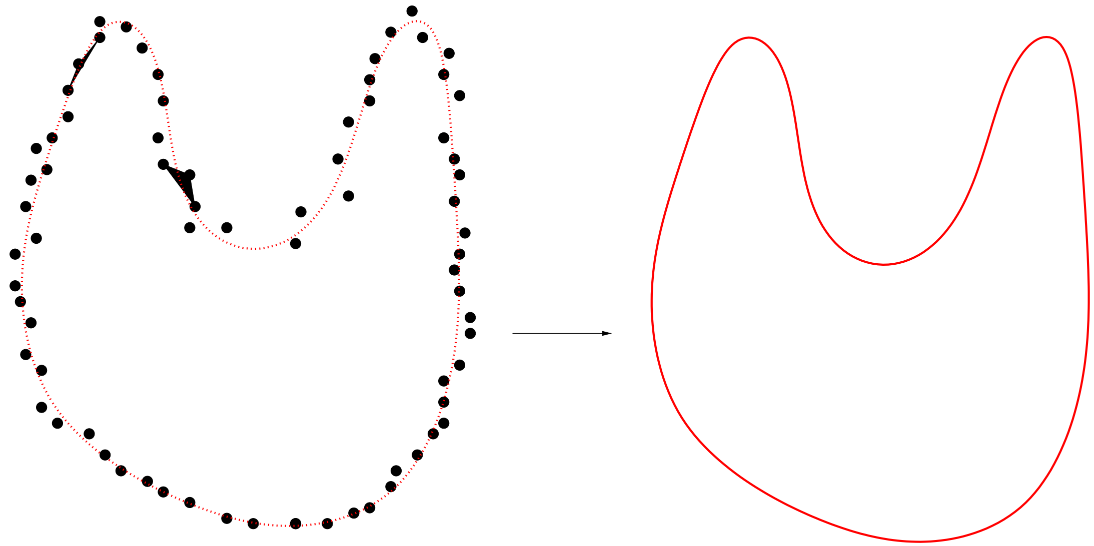
Joint work with Frédéric Chazal *(INRIA Saclay)*, David Cohen-Steiner *(INRIA Sophia-Antipolis)* and André Lieutier *(Dassault Systèmes)*,



Motivation

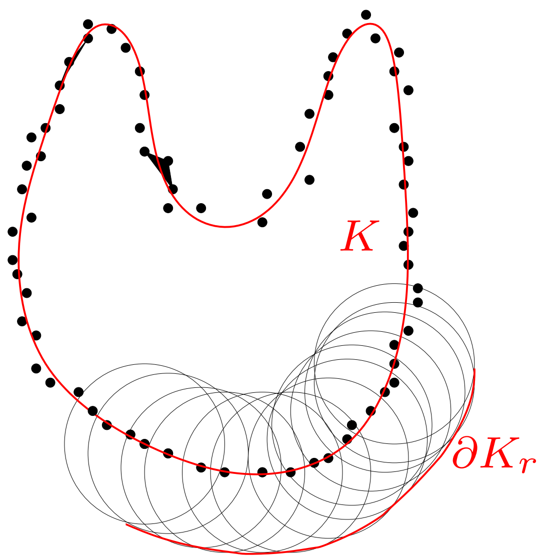
Certified geometric surface reconstruction



Given an Hausdorff approximation K' of a manifold K, can we estimate the geometric properties of K ? (and in particular the curvature of K)

$\longrightarrow$ The answer is of course “No".

Motivation



Let K and K' are two compact sets close for the Hausdorff distance. Then their offsets K'_r and K_r are also close.

$$K_r = \{x, d(x, K) \leq r\}$$

- Are the normals of $\partial K'_r$ and ∂K_r also close?
- Can we define the curvatures of $\partial K'_r$ and ∂K_r?
- If yes, are the curvatures of ∂K_r and $\partial K'_r$ close?

$\rightarrow$ Curvature measures
$\rightarrow$ A scale parameter

Previous results and contribution

- Stability results of the topology of the offsets (Grove, Chazal, Cohen-Steiner, Lieutier'06).
- Approximation of the curvatures of a smooth surface by the curvature measures of approximating triangulations. (Fu, Cohen-Steiner, Morvan).

Our contribution:

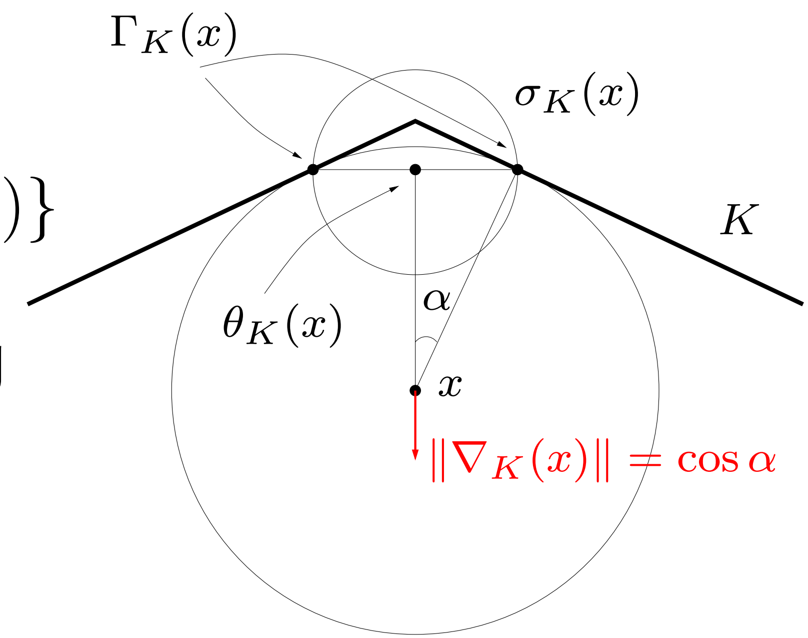
- we provide an explicit result of stability for the curvature measures of the offsets.

Remark A similar result of curvature measures stability has been obtained (F. Chazal, D. Cohen-Steiner, Q. Mérigot).

Overview

- Background on distance functions
- Definition of curvature measures
- Result of stability
- Curvature measures of 3D point clouds
- Sketch of Proof

The gradient of the distance function



$$\Gamma_K(x) = \{y \in K \mid d(x,y) = R_K(x)\}$$

$\sigma_K(x)$: smallest ball enclosing $\Gamma_K(x)$

$\theta_K(x)$: center of $\sigma_K(x)$

The generalized gradient:

$$\nabla_K(x) = \frac{x - \theta_K(x)}{R_K(x)}$$

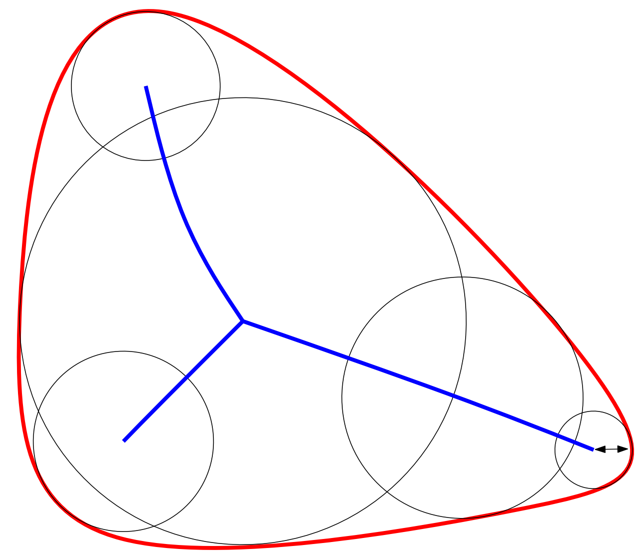
Medial axis and reach

Medial axis:

$$\mathcal{M}(K) = \{x \in \mathbb{R}^n \setminus K : \|\nabla_K(x)\| < 1\}$$

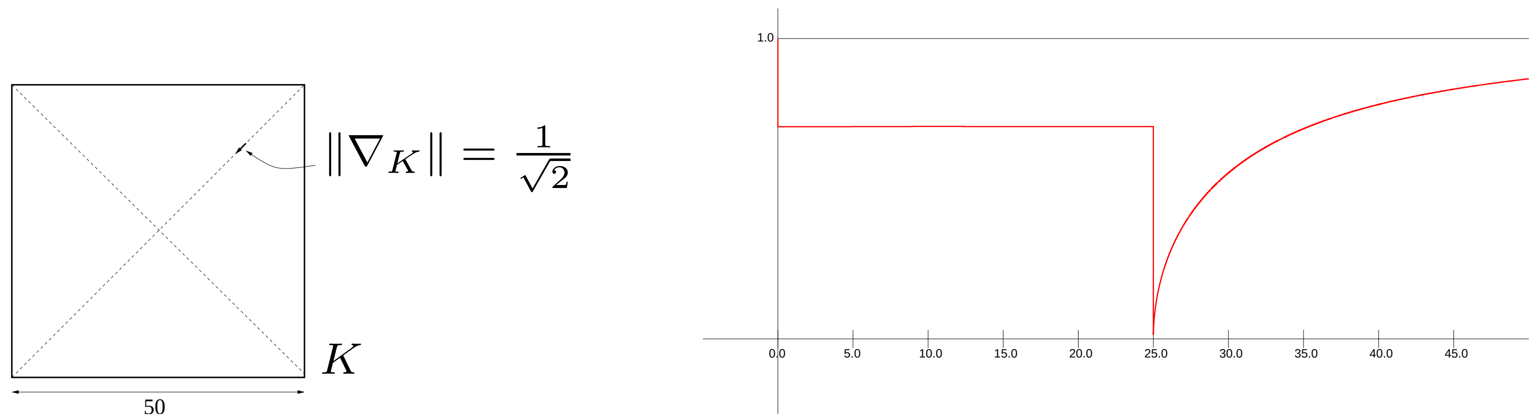
Offset:

$$K_r = \{x, d(x, K) \leq r\}$$



Reach$(K) = \sup\{r \geq 0 : K_r \cap \mathcal{M}(K) = \emptyset\}$

The critical function



The critical function of a square in 3D

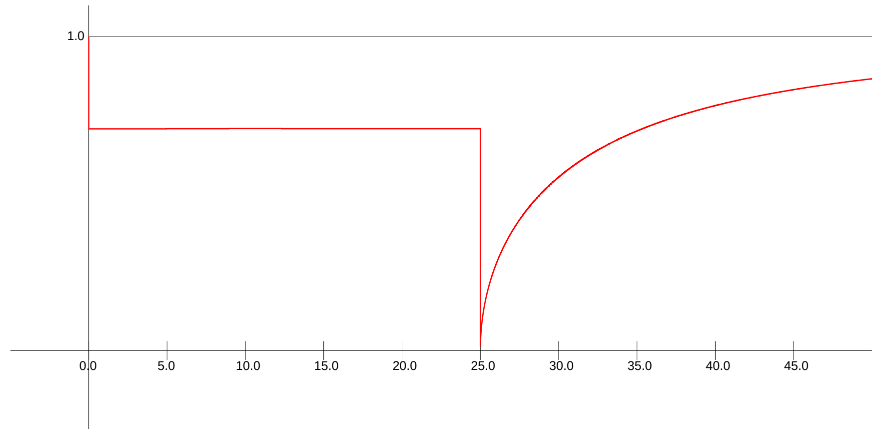
The critical function $\chi_K : (0, +\infty) \to \mathbb{R}_+$ is the real function defined by:

$$\chi_K(d) = \inf_{R_K^{-1}(d)} ||\nabla_K||$$

The μ-reach

$$r_\mu(K) = 25 \text{ if } \mu \leq \frac{1}{\sqrt{2}}$$

$$r_\mu(K) = 0 \text{ if } \mu > \frac{1}{\sqrt{2}}$$

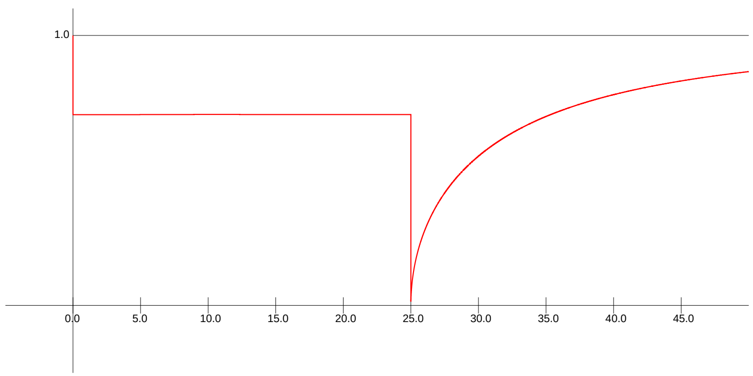


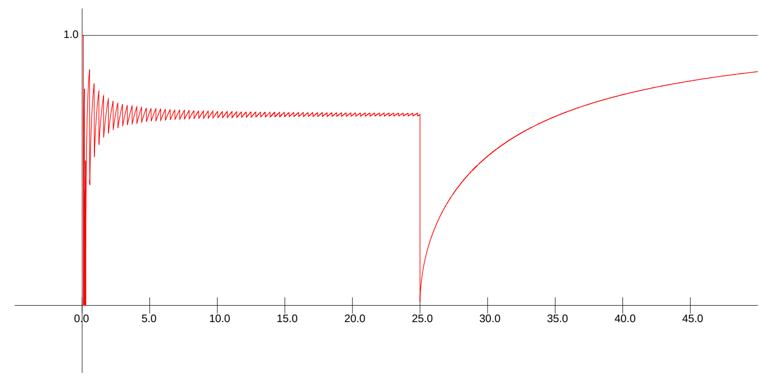
The *μ-reach* $r_\mu(K)$ of a compact set $K \subset \mathbb{R}^n$ is defined by:

$$r_\mu(K) = \inf\{d \mid \chi_K(d) < \mu\}$$

For $\mu = 1$, $r_1(K) = reach(K)$ is the reach of K (Federer).

Nice properties of reach and μ-*reach*



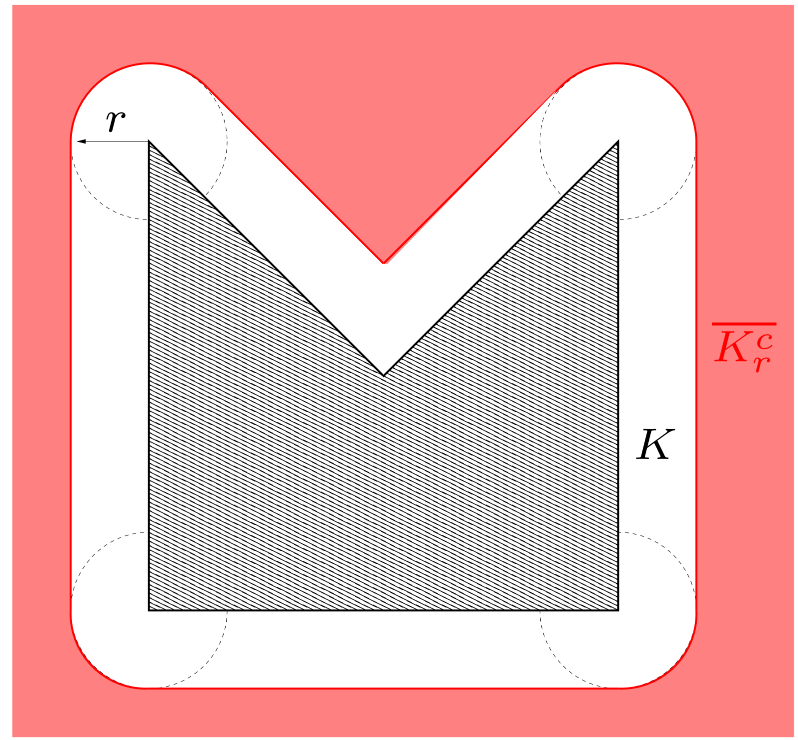


- Stability properties of the critical function allows to evaluate $r_\mu(K)$ from an approximation of K (C-Cohen-Steiner-Lieutier'06)
- If $r < reach(K)$ then the boundary of K_r is a smooth $\mathcal{C}^{1,1}$ hypersurface (Federer).

Reach of offsets complements

Let $K \subset \mathbb{R}^n$ be a compact set.
The offset K_r is given by:

$$K_r = \{x \in \mathbb{R}^n : R_K(x) \leq r\}$$



Theorem (Chazal-Cohen-Steiner-Lieutier-T'07)
For $r \in (0, r_\mu(K))$, one has $\mathrm{reach}(\overline{\mathrm{K_r^c}}) \geq \mu \mathrm{r}$.

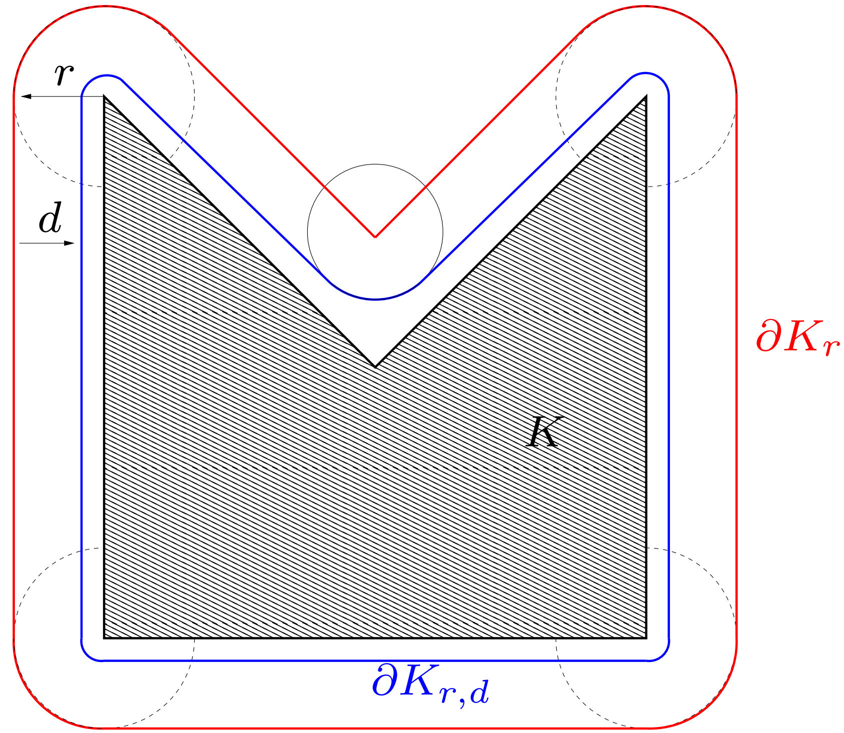
Smoothness of double offset

Offset:

$$K_r = \{x \in \mathbb{R}^n : R_K(x) \leq r\}$$

Double offsets:

$$K_{r,d} = \{x \in \mathbb{R}^n : d(x, K_r^c) \leq d\}$$



Theorem (Chazal-Cohen-Steiner-Lieutier-T'07)

If $r < r_\mu(K)$ and $d < \mu r$ then $\partial K_{r,d}$ is a smooth $\mathcal{C}^{1,1}$-hypersurface. Moreover, $\mathrm{reach}(\mathrm{K_{r,d}}) \geq \min(\mathrm{d}, \mu \mathrm{r} - \mathrm{d})$.

Overview

- Background on distance functions
- Definition of curvature measures
- Result of stability
- Curvature measures of 3D point clouds
- Sketch of Proof

Curvature measures of sets of reach > 0

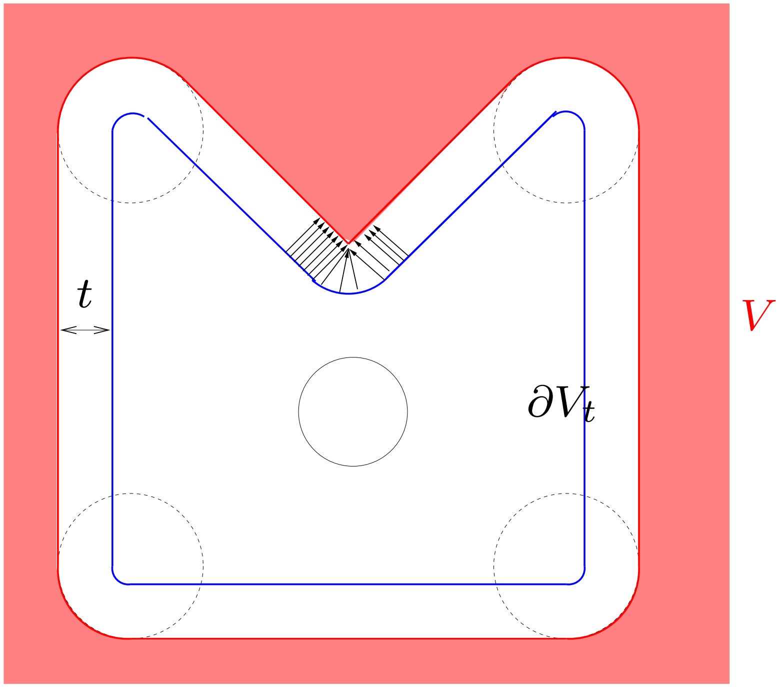
Let $V \subset \mathbb{R}^3$ be with positive reach.

$-$ $t < reach(V)$;

$-$ f is a Lipschitz function on $\mathbb{R}^3$;

$-$ G and H are the mean and gaussian curvature of ∂V_t

$-$ p_V is the projection onto V.



Definition

$$\Phi_V^G(f) = \lim_{t \to 0} \int_{\partial V_t} f(p_V(p))\, G(p) dp$$

$$\Phi_V^H(f) = \lim_{t \to 0} \int_{\partial V_t} f(p_V(p))\, H(p) dp,$$

$\longrightarrow$ can be generalised to the curvature measures $\Phi_V^i(f)$ in $\mathbb{R}^d$.

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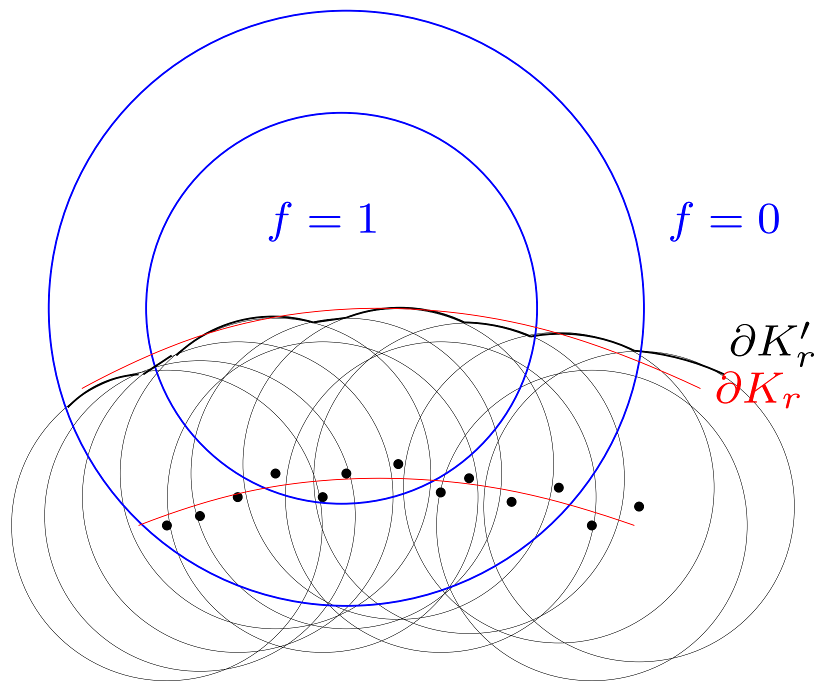
What do we compare ?



Is $\left|\Phi^i_{K_r}(f) - \Phi^i_{K'_r}(f)\right|$ small ?

Stability result

Theorem. Let K and K' be two compact sets of $\mathbb{R}^d$ whose μ-reaches are greater than r. Let $\epsilon = d_{Hauss}(K, K')$. If $\epsilon \leq \frac{r\mu\ (2-\sqrt{2})}{2} \min(\mu, \frac{1}{2})$, then for every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $|f| \leq 1$, one has:

$$|\Phi^i_{K_r}(f) - \Phi^i_{K'_r}(f)| \leq k(r, \mu, d, f)\ \sup(Lip(f), 1)\ \sqrt{\epsilon},$$

where $k(r, \mu, d, f)$ only depends on f through the covering number $\mathcal{N}(spt(f)_{O(\sqrt{\epsilon})}, \mu r/2)$; $Lip(f)$ is the Lipschitz-constant of f; $spt(f) = \{x \in \mathbb{R}^d,\ f(x) \neq 0\}$.

$\rightarrow$ Optimal upper bound.

$\rightarrow$ Same result for anisotropic curvature measures.

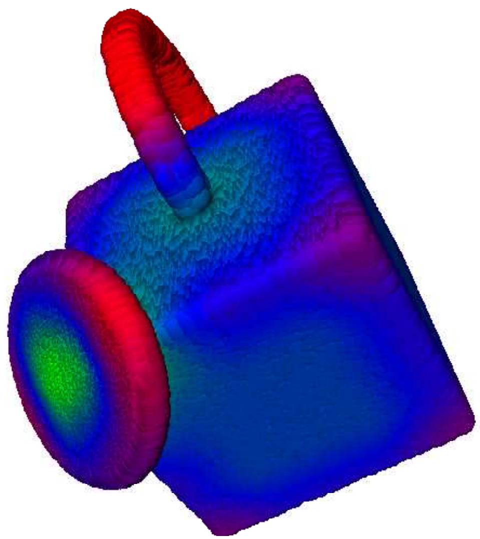
Stability result

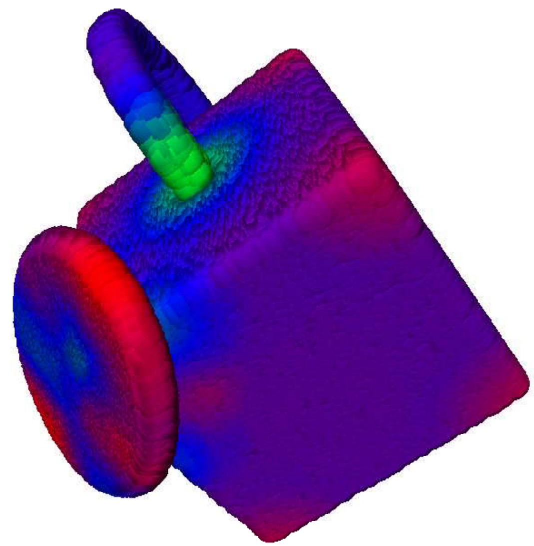
Theorem. Let K and K' be two compact subsets of $\mathbb{R}^d$ such that $r_\mu(K) > r$. Assume that the Hausdorff distance $\varepsilon = d_H(K, K')$ between K and K' is such that $\varepsilon < \frac{\mu^2}{60+9\mu^2} r$. Then the conclusions of the previous theorem also hold.

$\longrightarrow$ Can be applied to a compact set K with positive μ-reach that is approximated by a cloud of points K'.

Curvature measures of $3D$ point cloud

A point cloud P sampling a non manifold compact set.



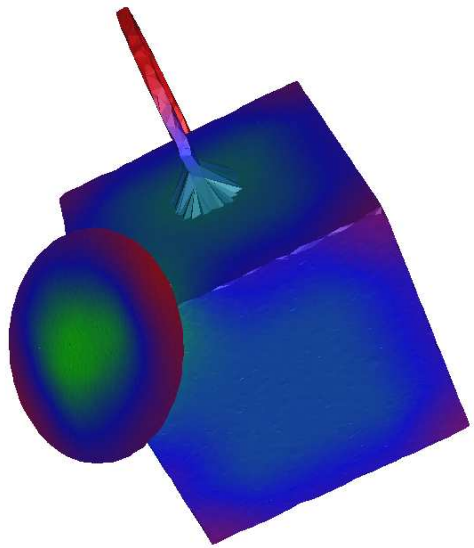


Mean curvature Gauss curvature

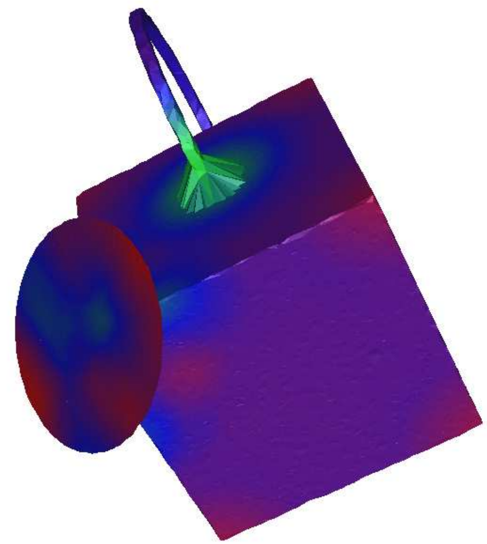
minimum value $\rightarrow$ in between $\rightarrow$ maximum value

Offset value $\alpha = 0.1$; diameter of the point cloud $= 2$; support of f contained in a ball of radius 0.3.

Curvature measures of $3D$ point cloud



Mean curvature



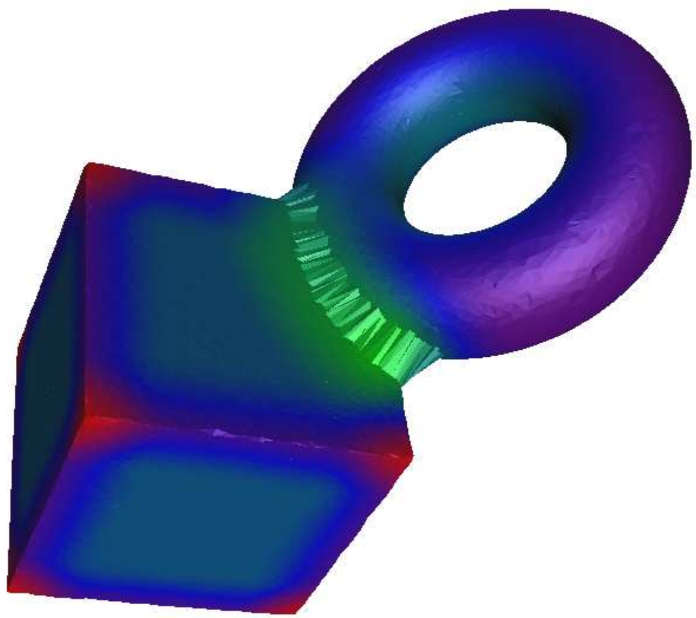
Gauss curvature

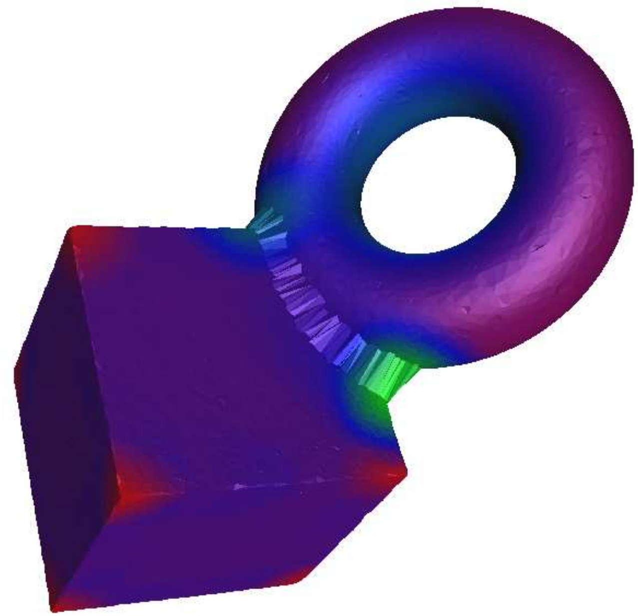
minimum value $\rightarrow$ in between $\rightarrow$ maximum value

One can also color the faces of the boundary of the α-shape of P

Curvature measures of $3D$ point cloud

Another example





Mean curvature Gauss curvature

minimum value $\longrightarrow$ in between $\longrightarrow$ maximum value

Overview

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- Definition of curvature measures
- Result of stability
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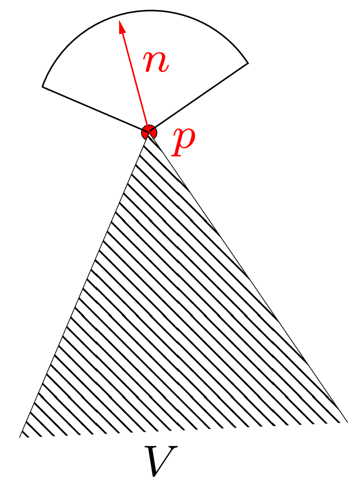
- Sketch of Proof (when we compare globally the offsets)

Definition

Let V be a set with positive reach r.



$S(V) =$
$\{(p, n) \in \mathbb{R}^d \times \mathcal{S}^{d-1}, p \in \partial V \;\; n \in CN(p)\}$,
where $CN(p)$ is the normal cone of V at p.

The normal cycle $N(V)$ of V is (d-1)-current on $\mathbb{R}^d \times \mathbb{R}^d$ (Fu'89):

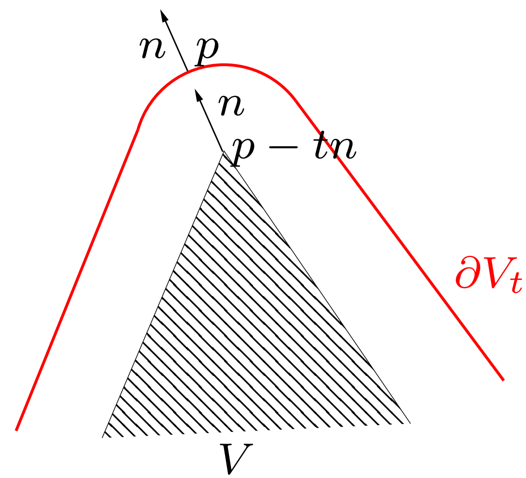
$$N(V) : \omega \mapsto \int_{S(V)} \omega.$$

$\rightarrow$ $N(V)$ contains all the curvature information.
$\rightarrow$ Example: in $\mathbb{R}^3$ $\Phi_V^H(f) = N(V)(\bar{f}\omega^H)$ with $\bar{f}(p, n) = f(p)$.

Step 1

Carrying the problem into the double offsets

$$F_{-t}: \quad \mathbb{R}^d \times \mathbb{R}^d \quad \longrightarrow \quad \mathbb{R}^d \times \mathbb{R}^d$$
$$(p, n) \quad \longmapsto \quad (p - tn, n)$$



If $t < reach(V)$, then $F_{-t} : spt(N(V_t)) \to spt(N(V))$ is one-to-one.

$$N(\overline{K_r}^c) - N(\overline{K_r'}^c) = F_{-t\sharp}(N(K_{r,t}) - N(K_{r,t})),$$

$\rightarrow$ Is $N(K_{r,t}) - N(K_{r,t})$ small ?

Step 2

Lemma In the condition of the theorem, and if $t \in (0, \mu r/2)$, one has:

$$d_H(\partial K_{r,t}, \partial K'_{r,t}) \leq \frac{\epsilon}{\mu},$$

$$\forall x \in K_{r,t} \quad 2\sin\frac{\angle\left(\nabla_{\overline{K_r{}^c}}(x), \nabla_{\overline{K'_r{}^c}}(x)\right)}{2} \leq 30\sqrt{\frac{\epsilon}{\mu t}}$$

That allows to show that

$$N(K_{r,t}) - N(K'_{r,t}) = \partial R,$$

where ∂R is the boundary of a particular d-rectifiable R current that satisfies : $\mathcal{H}^d(spt(R)) \leq O(\sqrt{\epsilon})$.

Step 3

One has for the mean curvature measure:

$$\begin{aligned}\Phi^H_{\overline{K_r}^c}(f) - \Phi^H_{\overline{K'_r}^c}(f) &= N(\overline{K_r{}^c})(\bar{f}\omega^H) - N(\overline{K'_r{}^c})(\bar{f}\omega^H) \\ &= F_{-t\sharp}\partial R(\bar{f}\omega^H).\end{aligned}$$

$$\begin{aligned}|F_{-t\sharp}\partial R(\bar{f}\omega^H)| &\leq (1+t^2)^{\frac{d-1}{2}}\ |\partial R(\bar{f}\omega^H)| \\ &\leq (1+t^2)^{\frac{d-1}{2}} \int_{spt(R)} d(\bar{f}\omega^H) \text{ (by Stockes'thm)} \\ &\leq 6\,(1+t^2)^{\frac{d-1}{2}}\,\mathcal{H}^d(spt(R))\sup(Lip(\bar{f}),1).\end{aligned}$$

$$\Phi^H_{K_r}(f) - \Phi^H_{K'_r}(f) = -(\Phi^H_{\overline{K_r}^c}(f) - \Phi^H_{\overline{K'_r}^c}(f))$$

$$\Rightarrow\ |\Phi^H_{K_r}(f) - \Phi^H_{K'_r}(f)| \leq k\,\sqrt{\epsilon}$$

Conclusion & Future works

- Isotropic and anisotropic curvature measures are Hausdorff stable.
- Pushing the normal cycle of K_r closer to K : a kind of α-normal cycle.