# Sur la topologie des courbes algébriques planes

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Travail conjoint avec J. Cheng, S. Lazard, M. Pouget, F. Rouillier and E. Tsigaridas

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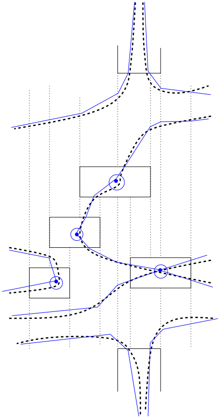
# Outline

# Topology and some geometry of real algebraic plane curves

Input curve: $f(x, y) = 0$ with $f \in \mathbb{Q}[x, y]$

- Isotopic approximation of the curve by a straight line graph
- give results in the original coordinate system of the plane
- In addition, identify and localize
  - extreme points,
  - singular points,
  - vertical asymptotes.



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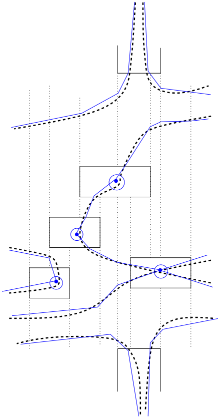

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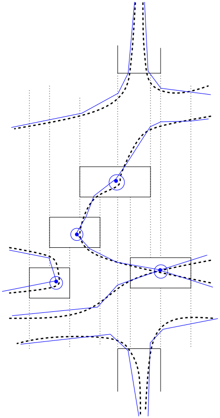

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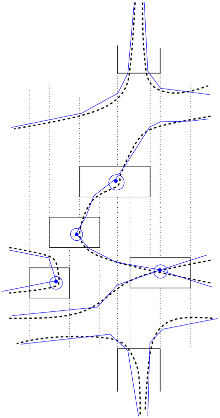

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Curve: square free polynomial $f \in \mathbb{Q}[x, y]$.

A point $\mathbf{p} = (\alpha, \beta) \in \mathbb{C}^2$ is (x-)critical if $f(\mathbf{p}) = f_y(\mathbf{p}) = 0$

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# Previous work

Mainly 2 approaches

## Subdivision

- only guarantee the drawing up to some precision
- need to go up to the theoretical separation bound to be certified
- or need to be coupled with an exact 2-D solver.

## Cylindrical Algebraic Decomposition-based with sub-resultant and lifting

Several variants:

- use Sturm-Habitch sequences or just principal S-H coefficients
- use generic position assumption
- use several projections
- shear and shear back

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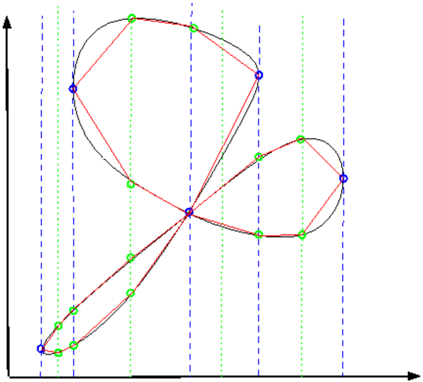
# CAD-based method

1. Projection
   Compute $x$-coordinates critical points: $\alpha_i$
2. Lifting
   Compute intersection points between the curve and the fiber $x = \alpha_i$
   Compute with polynomial with **algebraic** coefficients
3. Adjacencies
   Count the number of branches connected to the left and right
   May require generic position



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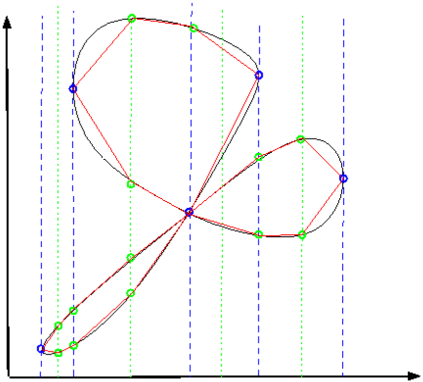

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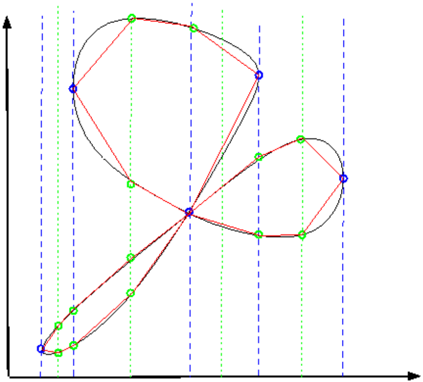
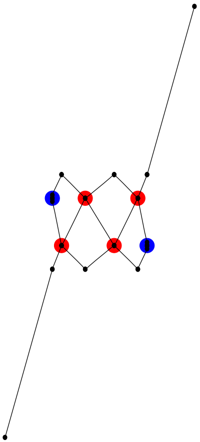


# General idea

- Replace sub-resultant sequences and computations with algebraic coefficient polynomials by
  Gröbner basis and Rational Univariate Representations
- Identify local topology at critical points using multiplicities and refinement
- Compute adjacencies with a vertical rectangular decomposition using multiplicities



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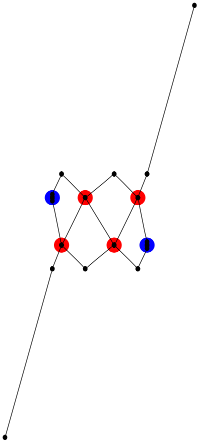

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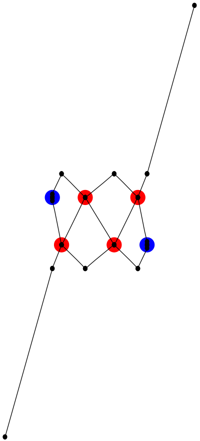

# Our algorithm

## Based on

- incremental work upon [WS05] and [CFPR08]
- Gröbner basis and Rational Univariate Representation of critical points.

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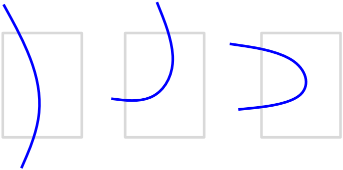

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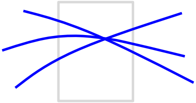

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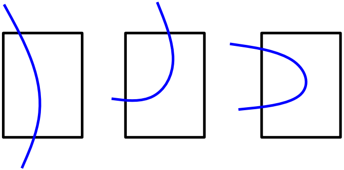

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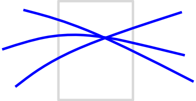

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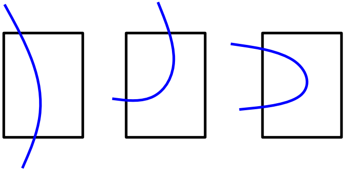

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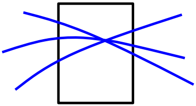

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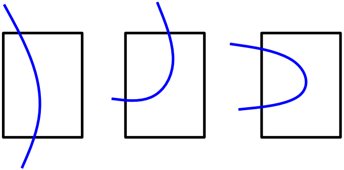

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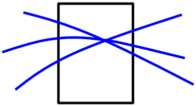

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$p_i \in \mathbb{K}[X]$

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$f + g \in \mathbb{I}$ if $f, g \in \mathbb{I}$
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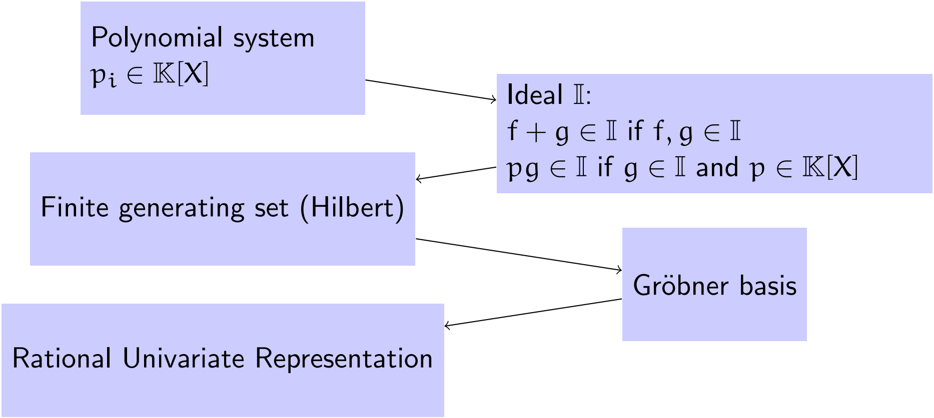


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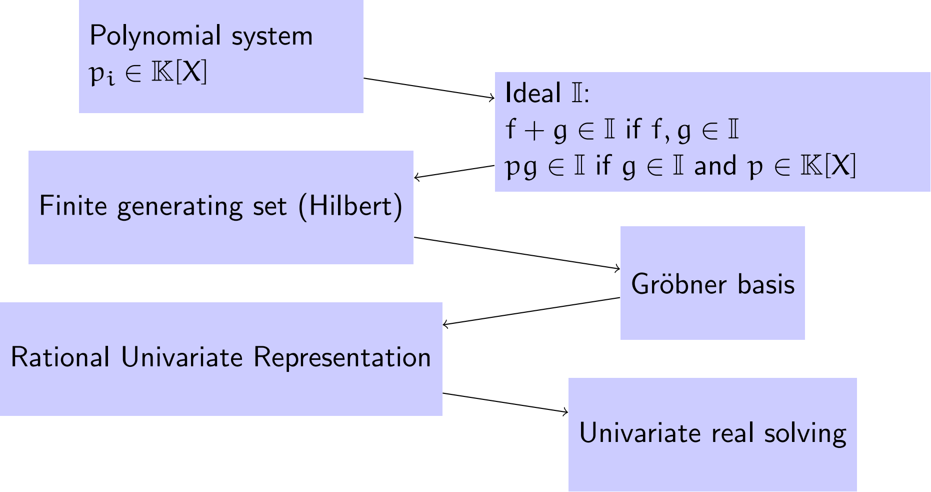

# Algebraic tools

- **Univariate** root isolation for polynomials with **rational** coefficients: **Descartes algorithm**
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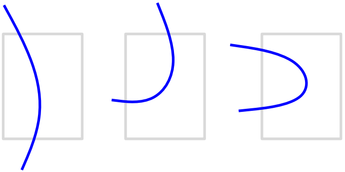
# Topology at extreme points

1. Isolate the extreme system
   $I_e = \mathbb{I}(f, f_y, f_x \neq 0) = \mathbb{I}(f, f_y, Tf_x - 1) \cap \mathbb{Q}[x, y]$
2. Refine boxes to get 2 crossings on the border



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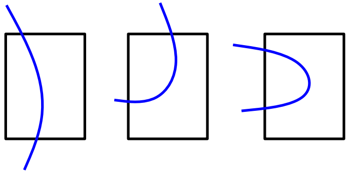

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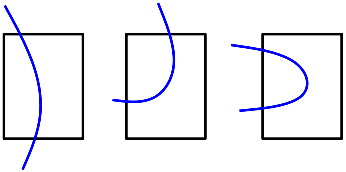

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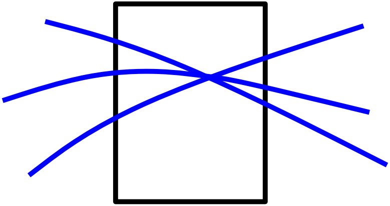
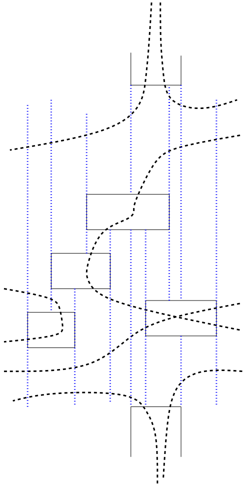
# Topology at singularities

1. Isolate singular points in boxes
2. Compute multiplicities $k$ in fibers
3. Refine the box to avoid the curve $f_{y^k} = \frac{\partial^k f}{\partial y^k}$
4. Refine the box to avoid top/bottom crossings



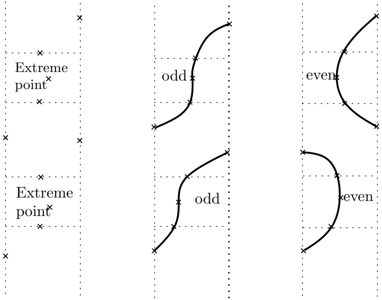
# Rectangle decomposition of the plane

- the topology is known inside critical boxes
- compute a vertical decomposition of the plane wrt these boxes
- compute intersections of the curve with the decomposition



# Greedy connection algorithm using multiplicities

Overlapping of extreme point boxes: need parity of multiplicity in fiber



# Complexity analysis

## Theorem

*The algorithm runs in* $\widetilde{\mathcal{O}}_B(Rd^4(d\tau s + s^2))$ *, where*

- $R$*: number of real critical points,*
- $d$*: degree of the polynomial* $f$*,*
- $\tau$*: maximum coefficient bitsize of* $f$*,*
- $s$*: maximum bitsize of*
  - *the separation bound of* $I_c$*, and*
  - *the distance between a singular point and its isolating curve (worst case* $s = d^3\tau$ *).*

# Implementation

Isotop: 7638 lines of Maple code using packages:

- FGB for Gröbner basis (Faugère)
- RS for RUR and isolation (Rouillier)

- faster on non generic and high degree curves
- robust
- exact

http://www.loria.fr/equipes/vegas/isotop/

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# Experiments

We ran large-scale tests, comparing to Brown's Cad2d, MPII's Alcix, González Vega's Top and Wolpert's Insulate.

| example index | $r = \frac{time_{\text{Cad2d}}}{time_{\text{ISOTOP}}}$ | example index | $r = \frac{time_{\text{Alcix}}}{time_{\text{ISOTOP}}}$ |
|---|---|---|---|
| 1 ... 114 | $1/43 \leq r < 1/3$ | 1 ... 17 | $1/5 < r < 1/3$ |
| 115 ... 382 | $1/3 \leq r \leq 1$ | 18 ... 33 | $1/3 \leq r \leq 1$ |
| 383 ... 479 | $1 < r \leq 3$ | 34 ... 290 | $1 < r \leq 3$ |
| 480 ... 513 | $3 \leq r \leq 160$ | 291 ... 465 | $3 \leq r \leq 64$ |
| 514 ... 585 | Cad2d timeout | 466 ... 585 | Alcix timeout |

# Conclusion

Summary of our contribution:

- deals with non generic case,
- gives results in the original coordinate system (identifies vertical asymptotes),
- avoids Sturm Habitch (principal coefficient),
- uses RUR,
- no restriction on singularities,
- enhancement for extreme points and the connection algorithm,
- output sensitive complexity analysis.