Random Tessellations and Polytopes

Rolf Schneider

U Freiburg

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Our topic: Some geometric questions about randomly generated mosaics in $\mathbb{R}^d$ and their typical polytopes

Typical questions in a nutshell (two historical problems):

(1) A question by Rényi in a lecture in Cambridge (1967):

Let L be a random line through the unit disc B^2 in the plane, with the following distribution:

The direction of the line (the angle with a fixed direction) is distributed uniformly in $[0, 2\pi]$; for given direction the oriented distance from 0 is distributed uniformly in $[-1, 1]$; direction and distance are independent.

Let $L_1, \ldots, L_n$ be stochastically independent random lines with the distribution of L. Let P_n be the intersection of the closed halfplanes bounded by these lines and containing 0.

Then P_n is a random polygon, and its vertex number $f_0(P_n)$ is a random variable. Rényi asked for its expectation, as n tends to infinity.

This was answered by Rényi and Sulanke (1968),

$$\lim_{n\to\infty} \mathbb{E}\, f_0(P_n) = \frac{\pi^2}{2} = 4.9348$$

Instead of fixing the unit disc and letting the number n tend to infinity, we shall consider infinitely many lines, spread out in the whole plane, and the generated tessellation.

Interesting questions arise for general distributions of the directions and in higher dimensions.

(2) A question by D.G. Kendall (1940s, popularized 1987):

Consider the random polygon P_n from (1). Under the condition that it has large area, will it be approximately circular, with high probability?

Actually, this question was posed for the cell containing 0 of the tessellation induced by certain random line systems (stationary isotropic Poisson line processes).

The answer is affirmative.

Again, the problem becomes more interesting for general directional distributions and in higher dimensions.

The answers to (1) and (2) depend heavily on results from the theory of convex bodies.

Preliminaries:

(a) Mosaics
(b) Poisson processes
(c) Amalgamation of both

(a) Mosaics

A mosaic in $\mathbb{R}^d$, or tessellation of $\mathbb{R}^d$, is a locally finite set m of d-polytopes in $\mathbb{R}^d$ with:

• $\bigcup_{P \in \mathsf{m}} P = \mathbb{R}^d$,

• for $P, Q \in \mathsf{m}$ with $P \neq Q$, $P \cap Q$ is either empty or a face of P and of Q.

The polytopes of m are called the cells of m. The k-faces of m are the k-faces of the cells of m.

$\mathcal{F}_k(\mathsf{m})$ is the set of all k-faces of m.

Examples

(1) Let $\mathcal{H}$ be a nonempty, locally finite set of hyperplanes in $\mathbb{R}^d$ (every compact subset of $\mathbb{R}^d$ meets only finitely many hyperplanes of $\mathcal{H}$).
The closures of the connected components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{H}} H$ form a mosaic. It is called the hyperplane tessellation induced by $\mathcal{H}$.

(2) Let $\mathcal{S}$ be a nonempty, locally finite set of points in $\mathbb{R}^d$. For $x \in \mathcal{S}$, the Voronoi cell (or Dirichlet cell) $C(x, \mathcal{S})$ of x (with respect to $\mathcal{S}$) is the set of all points in $\mathbb{R}^d$ for which x is a nearest point in $\mathcal{S}$, thus

$$C(x, \mathcal{S}) = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - s\| \text{ for all } s \in \mathcal{S}\}.$$

The point x is called the nucleus of the Voronoi cell $C(x, S)$.

$\{C(x, \mathcal{S}) : x \in \mathcal{S}\}$ is the Voronoi mosaic induced by $\mathcal{S}$.

(3) Suppose the points of $\mathcal{S}$ are in general position (no $d+1$ on a hyperplane, no $d+2$ on a sphere). Then each vertex of the Voronoi mosaic belongs to $d+1$ cells. The convex hull of the nuclei of these cells is a simplex. The set of all these simplices is the Delaunay mosaic induced by $\mathcal{S}$.

Direct definition: Any $d+1$ points of $\mathcal{S}$ lie on a sphere. If there is no point of $\mathcal{S}$ interior to this sphere, then the convex hull of the points is a cell of the Delaunay mosaic.

(b) Poisson processes

Let E be a locally compact topological space with a countable base. 'Measures' on E are Borel measures.

Let η be a locally finite subset of E.

We identify η with the (simple) counting measure on E that associates mass one to each element of η (and is zero else). Thus we use synonymously:

$$\begin{array}{rcl} x \in \eta & \text{and} & \eta(\{x\}) = 1, \\ \text{card}(\eta \cap A) & \text{and} & \eta(A). \end{array}$$

A point process in E is a measurable mapping from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the space of all simple counting measures on E (equipped with the smallest σ-algebra for which all evaluation maps $\eta \mapsto \eta(A)$, $A \subset E$ Borel set, are measurable).

Example: Let Θ be an infinite, locally finite measure on E without atoms. Let $C_i \subset E$ be compact sets with $\Theta(C_i) > 0$ and $C_i \uparrow E$ for $i \to \infty$. Let $m_i \in \mathbb{N}$, and let $\xi_1, \ldots, \xi_{m_i}$ be independent random points in E with distribution

$$\frac{\Theta \llcorner C_i}{\Theta(C_i)}.$$

Then $\{\xi_1, \ldots, \xi_{m_i}\}$ is a point process in E.

(This is often studied in $\mathbb{R}^d$ with Lebesgue measure and convex C_i. There are many papers about $\mathrm{conv}\{\xi_1, \ldots, \xi_{m_i}\}$.)

If we let i and m_i tend to infinity, in such a way that

$$\frac{m_i}{\Theta(C_i)} \to 1,$$

we obtain (heuristically) a Poisson process.

A (simple) point process X in E is a Poisson process with intensity measure Θ if

$$\mathbb{P}\{X(A) = k\} = \mathrm{e}^{-\Theta(A)} \frac{\Theta(A)^k}{k!} \qquad \text{for } k \in \mathbb{N}_0$$

for all Borel sets $A \subset E$ with $\Theta(A) < \infty$.

Properties:

- For disjoint Borel sets $A_1, A_2 \subset E$, the restrictions $X \llcorner A_1$ and $X \llcorner A_2$ are stochastically independent.

- If $A \subset E$ is a Borel set with $\Theta(A) < \infty$, then under the condition that $X(A) = m$, the restriction $X \llcorner A$ is stochastically equivalent to the process defined by m independent random points with distribution

$$\frac{\Theta \llcorner A}{\Theta(A)}.$$

(c) Random mosaics induced by Poisson processes

$\mathcal{K}^d$ denotes the space of convex bodies (nonempty, compact, convex subsets) in $\mathbb{R}^d$, equipped with the Hausdorff metric.

A process of convex particles in $\mathbb{R}^d$ is a point process X in the space $\mathcal{K}^d$.

It is called stationary (or homogeneous) if its distribution is invariant under translations, thus if, for each $t \in \mathbb{R}^d$, the processes X and $X + t$ are stochastically equivalent.

A random mosaic is a process of convex particles which is almost surely a mosaic.

We consider three types of random mosaics induced by Poisson processes.

(1) Poisson hyerplane mosaics

Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the scalar product and norm, respectively, in $\mathbb{R}^d$.

$B^d := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ is the unit ball and $S^{d-1} = \partial B^d$ the unit sphere.

A hyperplane in $\mathbb{R}^d$ can be written in the form

$$H(u, \tau) = \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\} = H(-u, -\tau),$$

with $u \in S^{d-1}$ and $\tau \in \mathbb{R}$.

Let $A(d, d-1)$ denote the space of all hyperplanes (affine Grassmannian) in $\mathbb{R}^d$, with its usual topology.

Let $\widehat{X}$ be a Poisson process in the space $A(d, d-1)$. Every realization of $\widehat{X}$ is a.s. a locally finite system of hyperplanes. It induces a tessellation of $\mathbb{R}^d$. In this way, a Poisson hyperplane mosaic X is defined.

(2) Poisson–Voronoi mosaics

Let $\widetilde{X}$ be a Poisson process in $\mathbb{R}^d$. Every realization of $\widetilde{X}$ is a.s. a locally finite system of points and hence defines its Voronoi tessellation. In this way, a Poisson–Voronoi mosaic X is defined.

(3) Poisson–Delaunay mosaics

Let $\widetilde{X}$ be a Poisson process in $\mathbb{R}^d$. Every realization of $\widetilde{X}$ is a.s. a locally finite system of points in general position and hence defines its Delaunay tessellation. In this way, a Poisson–Delaunay mosaic Y is defined.

In the following, $\widehat{X}$ and $\widetilde{X}$ are assumed to be stationary (their distributions are translation invariant). Then also the induced random mosaics are stationary.

Part I: Stationary Poisson hyperplane mosaics:
intersection densities and vertex numbers

Let $\widehat{X}$ be a stationary Poisson hyperplane process in $\mathbb{R}^d$.

It is determined by two parameters, a number and a probability measure on the sphere.

For a Borel set $A \in A(d, d-1)$, the random variable $\widehat{X}(A)$ is the number of hyperplanes of $\widehat{X}$ falling in A. Its expectation

$$\mathbb{E}\,\widehat{X}(A) = \Theta(A)$$

defines a measure on $A(d, d-1)$, the intensity measure of $\widehat{X}$.

By stationarity, there is a decomposition

$$\Theta(A) = \widehat{\gamma} \int_{S^{d-1}} \int_{-\infty}^{\infty} \mathbf{1}_A(H(u,\tau))\, \mathrm{d}\tau\, \widehat{\varphi}(\mathrm{d}u)$$

for Borel sets $A \subset A(d, d-1)$. Here,

$\widehat{\gamma}$ is the intensity of $\widehat{X}$,

$\widehat{\varphi}$ is the directional distribution of $\widehat{X}$, an even probability measure on the unit sphere.

We assume that $\widehat{\gamma} > 0$ and that $\widehat{\varphi}$ is not concentrated on a great subsphere.

Intuitive meaning of intensity and directional distribution:
For a symmetric Borel set $A \subset S^{d-1}$,

$$\widehat{\gamma}\widehat{\varphi}(A) = \frac{1}{2}\, \mathbb{E}\, \mathrm{card}\{H(u,\tau) \in \widehat{X} : u \in A,\ |\tau| \le 1\}.$$

Derived parameters: the intersection densities

Let $k \in \{0, \ldots, d-1\}$. Intersecting any $d-k$ hyperplanes of $\widehat{X}$ (which are a.s. in general position), we obtain a k-flat process, that is, a point process in the space $A(d,k)$ of k-flats of $\mathbb{R}^d$.

This is the intersection process of order $d-k$ of $\widehat{X}$, denoted by $\widehat{X}_{d-k}$.

Its intensity is given by

$$\widehat{\gamma}_{d-k} = \frac{1}{\kappa_{d-k}} \mathbb{E}\, \mathrm{card}\{F \in \widehat{X}_{d-k} : F \cap B^d \neq \emptyset\},$$

where κ_d is the volume of the d-dimensional unit ball.

It can also be represented by

$$\widehat{\gamma}_{d-k} = \mathbb{E} \sum_{F \in \widehat{X}_{d-k}} \mathrm{vol}_k(F \cap B),$$

for any Borel set $B \subset \mathbb{R}^d$ of Lebesgue measure $\lambda(B) = 1$.

We call $\widehat{\gamma}_{d-k}$ the intersection density of order $d-k$.

Note that $\widehat{X_1} = \widehat{X}$ and $\widehat{\gamma}_1 = \widehat{\gamma}$.

Let the intensity $\widehat{\gamma}$ be given. Then the hyperplane process $\widehat{X}$ depends only on the directional distribution $\widehat{\varphi}$.

This fact suggests many geometric extremal problems of isoperimetric type.

Some of them can be answered, using the device of the associated zonoid and results on convex bodies.

Which directional distributions yield 'the most intersections'?

More precisely, for given intensity $\widehat{\gamma}$, which directional distributions yield maximal intersection density of order $d-k$?

The intersection density of order $d-k$ is given by

$$\widehat{\gamma}_{d-k} = \frac{\widehat{\gamma}^{d-k}}{(d-k)!} \int_{(S^{d-1})^{d-k}} \nabla_{d-k}(u_1, \ldots, u_{d-k}) \, \widehat{\varphi}^{d-k}(\mathrm{d}(u_1, \ldots, u_{d-k})),$$

where $\nabla_{d-k}(u_1, \ldots, u_{d-k})$ is the $(d-k)$-dimensional volume of the parallelepiped spanned by the vectors $u_1, \ldots, u_{d-k}$.

The integral can be interpreted as an intrinsic volume of an auxiliary zonoid.

Explanations

The Minkowski sum of sets $A, B \subset \mathbb{R}^d$ is defined by

$$A + B = \{x + y : x \in A,\, y \in B\}.$$

The support function of a convex body K is defined by

$$h(K, u) = \max\{\langle x, u\rangle : x \in K\} \qquad \text{for } u \in \mathbb{R}^d.$$

It satisfies, $h(K+M, \cdot) = h(K, \cdot) + h(M, \cdot)$ for convex bodies K, M.

A sum of finitely many line segments is a zonotope.

The zonotopes are precisely the polytopes with centrally symmetric faces.

The support function of the segment with endpoints $v, -v$ is given by $h([v,-v],u) = |\langle u,v\rangle|$.

Hence, the support function of a zonotope K can be represented by

$$h(K,u) = \sum_{i=1}^{m} \alpha_i |\langle u, v_i\rangle|$$

with unit vectors $v_1, \ldots, v_m$ and positive numbers $\alpha_1, \ldots, \alpha_m$.

A zonoid is a convex body K whose support function has a representation

$$h(K,u) = \int_{S^{d-1}} |\langle u,v\rangle| \, \rho(\mathrm{d}v)$$

with a finite even measure ρ on the sphere S^{d-1}, or a translate of such a body. Equivalently, a convex body K is a zonoid if and only if it can be approximated by zonoids (in the Hausdorff metric).

The intrinsic volumes of a convex body K can be defined by the polynomial expansion

$$V(K + \epsilon B^d) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} V_j(K).$$

They satisfy the famous Aleksandrov–Fenchel inequalities, in particular

$$\left(\frac{\kappa_{d-j}}{\binom{d}{j}} V_j(K) \right)^k \geq \kappa_d^{k-j} \left(\frac{\kappa_{d-k}}{\binom{d}{k}} V_k(K) \right)^j$$

for $0 < j < k \leq d$. If $\dim K \geq j$, then equality holds if and only if K is a ball.

Application to our Poisson hyperplane process $\widehat{X}$:

Following Matheron, we define the associated zonoid $\Pi_{\widehat{X}}$ by its support function

$$h(\Pi_{\widehat{X}}, \cdot) = \frac{\widehat{\gamma}}{2} \int_{S^{d-1}} |\langle \cdot, v \rangle| \, \widehat{\varphi}(\mathrm{d}u).$$

Then it turns out that the intersection density of order k is given by

$$\boxed{\widehat{\gamma}_k = V_k(\Pi_{\widehat{X}})},$$

in particular, $\widehat{\gamma} = \widehat{\gamma}_1 = V_1(\Pi_{\widehat{X}})$. Therefore, the Aleksandrov–Fenchel inequality gives

$$\widehat{\gamma}_k \leq \frac{\binom{d}{k} \kappa_{d-1}^k}{d^k \kappa_{d-k} \kappa_d^{k-1}} \, \widehat{\gamma}^k,$$

for $k \in \{2, \dots, d\}$.

If equality holds, then $\Pi_{\widehat{X}}$ is a ball. By a known uniqueness theorem for the 'cosine transform', the even measure $\widehat{\varphi}$ must be the normalized spherical Lebesgue measure, hence it is rotation invariant. This implies that the intensity measure of the hyperplane process $\widehat{X}$ is rotation invariant.

For a Poisson process, the intensity measure determines the distribution. It follows that the distribution of $\widehat{X}$ is rotation invariant. A process with this property is called isotropic. The converse also holds. Hence:

For given intensity $\widehat{\gamma}$, the intersection density $\widehat{\gamma}_k$ of order $k \in \{2, \ldots, d\}$ becomes maximal precisely if $\widehat{X}$ is isotropic.

The preceding (older) result served us to introduce the associated zonoid $\Pi_{\widehat{X}}$.

This zonoid is now applied to more recent results.

Recall that X is the mosaic induced by the Poisson hyperplane process $\widehat{X}$. We denote by $X^{(k)}$ the process of its k-dimensional faces, $k = 2, \ldots, d$.

How many vertices has the 'average' k-face?

Here 'averaging' must be in two ways: spatially and stochastically.

We need a notion of a random polytope that can be considered as a 'typical' k-face of the mosaic X.

Two natural procedures, heuristically:

• Select a k-face at random, with equal weights for each k-face in a realization of the mosaic (which is, of course, only possible in a bounded region), and translate it so that its Steiner point comes to the origin. The resulting random polytope is called the typical k-face, denoted by $Z^{(k)}$.

• Select a random point p, uniformly with respect to the k-dimensional Hausdorff measure on the union of all k-faces (which is, of course, only possible in a bounded region), take the k-face containing p, and translate it by $-p$. The resulting random polytope is called the weighted typical k-face, denoted by $Z_0^{(k)}$.

Palm theory can be used to give precise definitions.

The zero cell Z_0 of X is the cell containing 0.
It is stochastically equivalent to $Z_0^{(d)}$.

Let $f_0(P)$ denote the number of vertices of a polytope P.

We ask for the expectations of the random variables

$$f_0(Z^{(k)}) \qquad \text{and} \qquad f_0(Z_0^{(k)}).$$

A (more general) theorem of J. Mecke (1984) gives

$$\mathbb{E}\, f_0(Z^{(k)}) = 2^k.$$

What about $\mathbb{E}\, f_0(Z_0^{(k)})$?

First we consider the case $k = d$, the expected vertex number of the zero cell Z_0.

One can compute (heavily using the Poisson assumption) that

$$
\begin{aligned}
&\mathbb{E}\, f_0(Z_0) \\
&= \frac{\widehat{\gamma}^d}{d!} \int_{(S^{d-1})^d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbb{E}\, \mathrm{card}(Z_0 \cap H(u_1, \tau_1) \cap \cdots \cap H(u_d, \tau_d)) \\
&\quad \times \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_d \, \widehat{\varphi}^d(\mathrm{d}(u_1, \ldots, u_d)) \\
&= \mathbb{E}\, V_d(Z_0) \cdot \frac{\widehat{\gamma}^d}{d!} \int_{(S^{d-1})^d} \nabla_d(u_1, \ldots, u_d)\, \widehat{\varphi}^d(\mathrm{d}(u_1, \ldots, u_d)) \\
&= \mathbb{E}\, V_d(Z_0)\, V_d(\Pi_{\widehat{X}}).
\end{aligned}
$$

Fortunately, also $\mathbb{E}\, V_d(Z_0)$ can be expressed in terms of the associated zonoid, namely

$$
\boxed{\mathbb{E}\, V_d(Z_0) = 2^{-d} d! V_d(\Pi^o_{\widehat{X}}).}
$$

Here, for a convex body K with 0 in the interior, K^o denotes the polar body, defined by

$$K^o = \{x \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } y \in K\}.$$

Thus, we obtain

$$\boxed{\mathbb{E}\, f_0(Z_0) = 2^{-d} d! V_d(\Pi_{\widehat{X}}) V_d(\Pi^o_{\widehat{X}}).}$$

The volume product $V_d(K)V_d(K^o)$ is an important functional in the affine geometry of convex bodies. For zonoids K (with centre 0), it satisfies the Reisner and Blaschke–Santaló inequalities

$$\frac{4^d}{d!} \leq V(K)V(K^o) \leq \kappa_d^2.$$

Equality on the left holds if and only if K is a parallelepiped, and on the right it holds if and only if K is an ellipsoid.

We conclude that

$$2^d \leq \mathbb{E} f_0(Z_0) \leq 2^{-d} d! \kappa_d^2.$$

On the left side, equality holds if and only if X is a parallel mosaic (that is, the hyperplanes of $\widehat{X}$ have only d fixed different directions). On the right side, equality holds if and only if X is affinely isotropic (that is, $\widehat{X}$ is an affine transform of an isotropic process).

We want to extend the preceding result to the weighted typical k-face $Z_0^{(k)}$, $k = 2, \dots, d$.

For this, we need a suitable representation of the weighted typical k-face $Z_0^{(k)}$ (that is, of its distribution).

Recall that the intersection process $\widehat{X}_{d-k}$ is obtained by intersecting any $d-k$ hyperplanes of the hyperplane process $\widehat{X}$ in general position. It is a stationary process of k-flats.

Its directional distribution $\widehat{\mathbb{Q}}_{d-k}$ is a probability measure on the Grassmannian $G(d,k)$ of k-dimensional linear subspaces.

Its intuitive meaning is exhibited by the following. For a k-flat F, let F_0 denote its translate through 0. For a Borel set $A \subset G(d,k)$ we have

$$\widehat{\mathbb{Q}}_{d-k}(A) = \frac{\mathbb{E}\,\mathrm{card}\{F \in \widehat{X}_{d-k} : F \cap B^d \neq \emptyset,\ F_0 \in A\}}{\mathbb{E}\,\mathrm{card}\{F \in \widehat{X}_{d-k} : F \cap B^d \neq \emptyset\}}.$$

The distribution of the weighted typical k-face is given by

$$\mathbb{P}(Z_0^{(k)} \in A) = \int_{G(d,k)} \mathbb{P}(Z_0 \cap L \in A)\, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L)$$

for Borel sets A in the space of polytopes.

In other words: the weighted typical k-face $Z_0^{(k)}$ is stochastically equivalent to the random polytope $Z_0 \cap \mathcal{L}$, where $\mathcal{L}$ is a random k-dimensional subspace of $\mathbb{R}^d$, independent of $\widehat{X}$, with distribution $\widehat{\mathbb{Q}}_{d-k}$.

For the expected number of vertices, this implies

$$\begin{aligned} \mathbb{E} f_0(Z_0^{(k)}) &= \int_{G(d,k)} \mathbb{E} f_0(Z_0 \cap L)\, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L) \\ &= \int_{G(d,k)} \mathbb{E} f_0(Z_0(\widehat{X} \cap L))\, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L). \end{aligned}$$

Here, $Z_0 \cap L$ is the intersection of the fixed subspace L with the random zero cell Z_0.

Equivalently, this is the zero cell of the intersection process $\widehat{X} \cap L$, which is a stationary Poisson hyperplane process in L. Hence, we can express $\mathbb{E} f_0(Z_0(\widehat{X} \cap L))$ in terms of the volume product of the associated zonoid of $\widehat{X} \cap L$.

Fortunately (Matheron), the associated zonoid of the intersection process $\widehat{X} \cap L$ is the orthogonal projection of the associated zonoid of $\widehat{X}$ to L, denoted by $\Pi_{\widehat{X}}|L$. Hence,

$$\mathbb{E} f_0(Z_0^{(k)}) = 2^{-k} k! \int_{G(d,k)} V_k(\Pi_{\widehat{X}}|L) V_k((\Pi_{\widehat{X}}|L)^o) \, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L).$$

In each subspace L, we can apply the Reisner and the Blaschke–Santaló inequality. This gives

Theorem. *For $k = 2, \ldots, d$,*

$$2^k \leq \mathbb{E} f_0(Z_0^{(k)}) \leq 2^{-k} k! \kappa_k^2.$$

On the left side, equality holds if and only if X is a parallel mosaic. On the right side, equality holds if X is isotropic.

Before discussing the equality cases, we mention a generalization. The distribution of the weighted typical k-face $Z_0^{(k)}$ is (if translations are disregarded) the volume weighted distribution of the typical k-face $Z^{(k)}$. We can also use different weighting functions, namely

$$L_j(P) = \mathcal{H}^j(\mathrm{skel_j} P) = \sum_{F \in \mathcal{F}_j(P)} V_j(F),$$

for polytopes P and for $0 \leq j \leq \dim P$.

The L_j-weighted typical k-face $Z_{k,j}$ has distribution given by

$$\mathbb{P}(Z_{k,j} \in A) = \frac{1}{\mathbb{E}\, L_j(Z^{(k)})} \mathbb{E}\left[\mathbf{1}_A(Z^{(k)}) L_j(Z^{(k)})\right].$$

Then, for $0 \le j \le k \le d$,

$$2^k \le \mathbb{E} f_0(Z_{k,j}) \le 2^{j-2k} \sum_{i=0}^{k-j} 2^{2i} \binom{k-j}{i} (k-i)! \kappa_{k-i}^2.$$

Equality cases are as before.

The equality cases

They pose geometric questions on projections of zonoids.

From

$$\mathbb{E}f_0(Z_0^{(k)}) = 2^{-k}k! \int_{G(d,k)} V_k(\Pi_{\widehat{X}}|L)V_k((\Pi_{\widehat{X}}|L)^o)\, \widehat{\mathbb{Q}}_{d-k}(\mathrm{d}L)$$

and

$$\frac{4^k}{k!} \leq V_k(\Pi_{\widehat{X}}|L)V_k((\Pi_{\widehat{X}}|L)^o) \leq \kappa_k^2$$

we have obtained that

$$2^k \leq \mathbb{E}f_0(Z_0^{(k)}) \leq 2^{-k}k!\kappa_k^2. \tag{1}$$

Equality on the left side of (1) holds if and only if the projection $\Pi_{\widehat{X}}|L$ is a parallelepiped for each subspace L in the support of the measure $\widehat{\mathbb{Q}}_{d-k}$.

Explicitly,

$$\begin{aligned}\widehat{\mathbb{Q}}_{d-k}(A) &= c(\widehat{\varphi}) \int_{(S^{d-1})^{d-k}} \mathbf{1}_A(u_1^\perp \cap \cdots \cap u_{d-k}^\perp) \nabla_{d-k}(u_1, \ldots, u_{d-k}) \\ &\quad \times \widehat{\varphi}^{d-k}(\mathrm{d}(u_1, \ldots, u_{d-k}))\end{aligned}$$

for Borel sets $A \subset G(d,k)$. Recall that

$$h(\Pi_{\widehat{X}}, x) = \frac{\widehat{\gamma}}{2} \int_{S^{d-1}} |\langle u, x \rangle| \, \widehat{\varphi}(\mathrm{d}u), \qquad x \in \mathbb{R}^d.$$

If $\Pi_{\widehat{X}}$ is a parallelepiped, then $\widehat{\varphi}$ is concentrated in $\pm e_1, \ldots, \pm e_d$, for a basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$.

$\Pi_{\widehat{X}}$ is the sum of segments parallel to $e_1, \ldots, e_d$.

Let $L \in \operatorname{supp} \widehat{\mathbb{Q}}_{d-k}$.

Then $L = u_1^\perp \cap \cdots \cap u_{d-k}^\perp$ with linearly independent vectors in the support of $\widehat{\varphi}$, say $(u_1, \dots, u_{d-k}) = (e_1, \dots, e_{d-k})$, hence $L = \operatorname{lin}\{e_1, \dots, e_{d-k}\}^\perp$. The projection $\Pi_{\widehat{X}}|L$ is the sum of segments parallel to the projections of $e_{d-k+1}, \dots, e_d$, hence it is a parallelepiped.

Conversely, suppose that $\Pi_{\widehat{X}}|L$ is a parallelepiped, for each $L \in \operatorname{supp} \widehat{\mathbb{Q}}_{d-k}$. We have to show that $\Pi_{\widehat{X}}$ is a parallelepiped.

Sketch of the proof for $k = d - 1$

We have a zonoid K with generating measure ρ, thus

$$h(K, u) = \int_{S^{d-1}} |\langle u, v \rangle| \, \rho(\mathrm{d}u), \qquad u \in S^{d-1}.$$

For each $e \in \operatorname{supp} \rho$, the projection $K|e^\perp$ is a parallelepiped.

For $e \in S^{d-1}$, the projection $K|e^\perp$ is a zonoid with generating measure $\pi_{e^\perp}\rho$, given by

$$\pi_{e^\perp}\rho = \int_{S^{d-1}\setminus\{\pm e\}} \mathbf{1}_A\left(\frac{u|e^\perp}{\|u|e^\perp\|}\right)\|u|e^\perp\|\,\rho(\mathrm{d}u)$$

for Borel sets $A \subset S^{d-1} \cap e^\perp$.

For simplicity, let $d = 3$. Let $e \in \operatorname{supp}\rho$. Then $K|e^\perp$ is a parallelogram, hence its generating measure is concentrated in two pairs of antipodal points. Therefore, ρ is concentrated on two great circles C_1, C_2 through e. Choose $e' \in C_1 \cap \operatorname{supp}\rho \setminus \{\pm e\}$. Then ρ is concentrated on two great circles C_3, C_4 through e'. One of them must conicide with C_1, say C_4. Then ρ is concentrated in the intersection points of C_1, C_2, C_3. Hence, K is a parallelepiped.

Part II: The shape of large cells

For a set η of hyperplanes in $\mathbb{R}^d$, not containing 0, let

$$Z_0 := \bigcap_{H\in\eta} H^-,$$

where H^- is the closed halfspace bounded by H that contains 0.

We study this for a Poisson process $\widehat{X}$ of hyperplanes. Then Z_0 is a random polytope.

In order to cover stationary Poisson hyperplane tessellations as well as stationary Poisson–Voronoi mosaics, we assume a more general form of the intensity measure

$$\Theta = \mathbb{E}\,\widehat{X}(\cdot).$$

For Borel sets $A \subset A(d, d-1)$, let

$$\Theta(A) = \lambda \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H(u,\tau))\, \tau^{r-1}\, \mathrm{d}\tau\, \varphi(\mathrm{d}u).$$

Here,

$\lambda > 0$	is an intensity parameter,
$r \geq 1$	is the distance exponent,
φ	is the directional distribution, a probability measure on S^{d-1}, not concentrated on a closed hemisphere.

Question: What is the asymptotic shape of the zero cell

$$Z_0 := \bigcap_{H \in \widehat{X}} H^-$$

under the condition that it is large, in some sense?

Program:

The asymptotic shape of large cells is completely determined by the equality cases of an isoperimetric inequality for convex bodies.

Stability results for the isoperimetric inequality yield estimates for the deviation of large cells from limit shapes.

Origin:

In the early 1940s, David G. Kendall had a conjecture about the zero cell Z_0 of a stationary, isotropic Poisson process of lines in the plane. In the book

D. Stoyan, W.S. Kendall, J. Mecke: *Stochastic Geometry and its Applications* (1987)

he repeated his conjecture in the foreword:

"One would have preferred to be able to say something about . . . my conjecture that the conditional law for the shape of Z_0, given the area $A(Z_0)$ of Z_0, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the circular shape."

What happened in the meantime:

R.E. Miles 1995: A heuristic proof of a longstanding conjecture of D.G. Kendall concerning the shapes of certain large random polygons

I.N. Kovalenko 1997: A proof of a conjecture of David Kendall on the shape of random polygons of large area

I.N. Kovalenko 1998: An extension of a conjecture of D.G. Kendall concerning shapes of random polygons to Poisson Voronoï cells

J. Mecke, I. Osburg 2003: On the shape of large Crofton parallelotopes

D. Hug, M. Reitzner, R. Schneider 2004 a,b
D. Hug, R. Schneider 2004, 2005, 2007 a,b

Extension of Kendall's problem in various directions:

(a) higher dimensions, that is, stationary, isotropic Poisson hyperplane processes and zero cells of large volume,

(b) measuring the size not only by the volume, but also by other functionals,

(c) dropping the assumption of isotropy (rotation invariance)

(d) dropping the stationarity (translation invariance)

The latter opens the way to including typical cells of Poisson–Voronoi tessellations.

Poisson–Voronoi tessellations

Let $\mathcal{S}$ be a locally finite set in $\mathbb{R}^d$. Recall that, for $x \in \mathcal{S}$, the set

$$C(x, \mathcal{S}) := \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - s\| \ \ \forall\, s \in \mathcal{S}\},$$

consisting of all points of $\mathbb{R}^d$ for which x is a nearest point in $\mathcal{S}$, is the Voronoi cell (or Dirichlet cell) of x with respect to $\mathcal{S}$.

Let $\widetilde{X}$ be a stationary Poisson point process in $\mathbb{R}^d$. Then

$$\{C(x, \widetilde{X}) : x \in \widetilde{X}\}$$

is a stationary random tessellation, called the Poisson–Voronoi tessellation induced by $\widetilde{X}$.

Its typical cell is (by Slivnyak's theorem) stochastically equivalent to

$$C(0, \widetilde{X} \cup \{0\}).$$

Thus, the typical cell can be defined by

$$Z = \bigcap_{x \in \widetilde{X}} H(x)^-,$$

where $H(x)$ is the mid hyperplane of 0 and x.

Thus, the typical cell Z of the Poisson–Voronoi tessellation is the zero cell of the hyperplane mosaic generated by the hyperplane process

$$\{H(x) : x \in \widetilde{X}\}.$$

This is a nonstationary Poisson hyperplane process.

Forms of the intensity measure:

(a) For a stationary Poisson hyperplane process:

$$\Theta(A) = 2\widehat{\gamma}\int_{S^{d-1}}\int_0^\infty \mathbf{1}_A(H(u,\tau))\,\mathrm{d}\tau\,\widehat{\varphi}(\mathrm{d}u),$$

$\widehat{\gamma}$ the intensity, $\widehat{\varphi}$ the directional distribution

(b) For the mid hyperplanes of 0 and a stationary Poisson point process of intensity γ:

$$\Theta(A) = 2^d\gamma\int_{S^{d-1}}\int_0^\infty \mathbf{1}_A(H(u,\tau))\tau^{d-1}\,\mathrm{d}\tau\,\sigma(\mathrm{d}u),$$

σ the rotation invariant probability measure on the sphere S^{d-1}.

Common generalization: a Poisson hyperplane process with intensity measure

$$\Theta(A) = \lambda\int_{S^{d-1}}\int_0^\infty \mathbf{1}_A(H(u,\tau))\tau^{r-1}\,\mathrm{d}\tau\,\varphi(\mathrm{d}u).$$

Intermediate résumé:

Given: a Poisson hyperplane process $\widehat{X}$ with intensity measure

$$\Theta(A) = \lambda \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H(u,\tau))\tau^{r-1}\, \mathrm{d}\tau\, \varphi(\mathrm{d}u)$$

and its zero cell

$$Z_0 := \bigcap_{H \in \widehat{X}} H^-,$$

Given: a functional Σ to measure the size of Z_0.

Question: What is the asymptotic shape of Z_0, under the condition that the size $\Sigma(Z_0)$ tends to infinity?

Does it exist?
What is a candidate for the shape?
How does it depend on Σ?

The Size Functional:

How ‘large’ the zero cell is, can be measured by any real function Σ on $\mathcal{K}^d$ (the space of convex bodies in $\mathbb{R}^d$) which is

- increasing under set inclusion,
- homogeneous of some degree $k \geq 0$,
- continuous,
- $\not\equiv 0$.

We call Σ a size functional.

Examples: volume, surface area, mean width, diameter, thickness, inradius, circumradius, volume of the John ellipsoid, width in a given direction, . . .

Crucial is the relation of the size functional to

The Hitting Functional:

For a convex body $K \in \mathcal{K}^d$, define

$$\mathcal{H}_K := \{H \in A(d, d-1) : H \cap K \neq \emptyset\}.$$

By the Poisson distribution of X and the form of the intensity measure,

$$\mathbb{P}(\mathrm{card}(X \cap \mathcal{H}_K) = k) = \frac{[\Phi(K)\lambda]^k}{k!}\, \mathrm{e}^{-\Phi(K)\lambda}$$

with

$$\Phi(K) = \frac{1}{r} \int_{S^{d-1}} h(K, u)^r \, \varphi(\mathrm{d}u).$$

We call Φ the hitting functional.

The Isoperimetric Inequality:

By continuity and homogeneity, the hitting functional Φ and the size functional Σ satisfy a sharp isoperimetric inequality

$$\Phi(K) \geq \alpha \Sigma(K)^{r/k}. \tag{2}$$

'Sharp' means that there are extremal bodies (with more than one point) for which equality holds (this determines α).

We need a stability version, quantifying that a body which almost yields equality must be close to an extremal body.

The Deviation Functional:

For given Φ and Σ, let ϑ be a function on $\{K \in \mathcal{K}^d : \Sigma(K) > 0\}$ with the following properties:

- $\vartheta(K) = 0$ for $K \in \mathcal{K}^d \iff K$ is extremal,
- ϑ is continuous,
- nonnegative,
- homogeneous of degree zero.

We call ϑ a deviation functional.

Deviation functionals exist.

Stability Version of the Isoperimetric Inequality:

There exists a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(\epsilon) > 0$ for $\epsilon > 0$ and $f(0) = 0$ such that

$$\vartheta(K) \geq \epsilon \quad \Rightarrow \quad \Phi(K) \geq (1 + f(\epsilon))\alpha\Sigma(K)^{r/k}, \tag{3}$$

for $K \in \mathcal{K}^d$.

Any such f is called a stability function for Φ, Σ, ϑ.

In concrete cases, explicit stability functions are of interest.

Heuristic Approach

We want an upper estimate for the conditional probability

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) = \frac{\mathbb{P}(\vartheta(Z_0) \geq \epsilon, \Sigma(Z_0) \geq a)}{\mathbb{P}(\Sigma(Z_0) \geq a)}.$$

Let B be an extremal body, with $0 \in B$ and $\Sigma(B) = a$; then $\Phi(B) = \alpha \Sigma(B)^{r/k} = \alpha a^{r/k}$.

Denominator: By the monotonicity of Σ,

$$\mathbb{P}(\Sigma(Z_0) \geq a) \geq \mathbb{P}(\mathrm{card}(X \cap \mathcal{H}_B) = 0) = \exp\left\{-\alpha a^{r/k} \lambda\right\}.$$

Numerator: Let a (deterministic) convex body K satisfy

$$\vartheta(K) \geq \epsilon > 0, \qquad \Sigma(K) \geq a.$$

Then, by the stability version (3) of the isoperimetric inequality,

$$\begin{aligned}
\mathbb{P}(\mathrm{card}(X \cap \mathcal{H}_K) = 0) &= \exp\{-\Phi(K)\lambda\} \\
&\leq \exp\left\{-(1 + f(\epsilon))\alpha\Sigma(K)^{r/k}\lambda\right\} \\
&\leq \exp\left\{-(1 + f(\epsilon))\alpha a^{r/k}\lambda\right\}.
\end{aligned}$$

Heuristically, we replace the deterministic body K satisfying

$$\text{card}(X \cap \mathcal{H}_K) = 0, \qquad \vartheta(K) \geq \epsilon, \qquad \Sigma(K) \geq a$$

by the random body Z_0 satisfying

$$\text{card}(X \cap \mathcal{H}_{\beta Z_0}) = 0 \;\; \forall\, \beta \in (0,1), \quad \vartheta(Z_0) \geq \epsilon, \quad \Sigma(Z_0) \geq a,$$

and hope that this costs only a slight weakening of the inequality, say

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon, \Sigma(Z_0) \geq a) \leq c_1 \exp\left\{-(1 + c_2 f(\epsilon))\alpha a^{r/k}\lambda\right\}.$$

Division then gives the exponential estimate

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c_1 \exp\left\{-c_2 f(\epsilon)\alpha a^{r/k}\lambda\right\}.$$

This can be made precise.

Theorem 1. *With a suitable constant $c_0 > 0$ (depending only on α), the following holds. If $\epsilon > 0$ and $a > 0$, then*

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c \exp\left\{-c_0 f(\epsilon) a^{r/k} \lambda\right\} \qquad (4)$$

where c is a constant depending on $\varphi, r, \Sigma, f, \epsilon$, but not on a or λ.

This shows that the extremal bodies of the isoperimetric inequality (2) can be considered as asymptotic shapes.

But Kendall's problem asked for limit shapes.

Shapes

Let G be one of the groups of: similarities, homotheties, positive dilatations of $\mathbb{R}^d$.

The G-shape of $K \in \mathcal{K}^d$ is the orbit $s_{\mathsf{G}}(K) := \{gK : g \in \mathsf{G}\}$. Let $\mathcal{S}_{\mathsf{G}}$ be the space of all G-shapes, with the quotient topology.

The conditional law of the G-shape of Z_0, given the lower bound $a > 0$ for the size $\Sigma(Z_0)$, is the probability measure μ_a on $\mathcal{S}_{\mathsf{G}}$ defined by

$$\mu_a(A) := \mathbb{P}(s_{\mathsf{G}}(Z_0) \in A \mid \Sigma(Z_0) \geq a)$$

for Borel sets $A \subset \mathcal{S}_{\mathsf{G}}$.

Theorem 2. *Suppose that the extremal bodies of* (2) *belong to a unique* G*-shape* $s_{\mathsf{G}}(B)$*. Then* $s_{\mathsf{G}}(B)$ *is the limit shape of* Z_0 *for increasing* Σ*, in the sense that*

$$\lim_{a\to\infty} \mu_a = \delta_{s_{\mathsf{G}}(B)} \quad \textit{weakly},$$

where $\delta_{s_{\mathsf{G}}(B)}$ *denotes the Dirac measure concentrated at* $s_{\mathsf{G}}(B)$*.*

These are two abstract, general theorems. To apply them in concrete cases (special Θ and Σ), one has to find the exact isoperimetric inequality (2) and its extremal bodies (and, preferably, explicit stability functions).

Some Examples:

(1) The zero cell of a stationary Poisson hyperplane tessellation; the size measured by the volume. Here,

$$\Phi(K) = \int_{S^{d-1}} h(K,u)\,\varphi(\mathrm{d}u) = dV(K,B,\ldots,B),$$

where B is the centered convex body with area measure φ (it exists by Minkowski's theorem).

The crucial isoperimetric inequality is Minkowski's inequality

$$V(K,B,\ldots,B) \geq V_d(B)^{1-1/d} V_d(K)^{1/d}.$$

Equality holds if and only if K is homothetic to B. Hence, the homothety class of B is the limit shape of Z_0 with respect to the volume. (Stability: Groemer)

(2) The typical cell of a stationary Poisson–Voronoi tessellation; the size measured by the kth intrinsic volume V_k. Here,

$$\Phi(K) = \frac{1}{d} \int_{S^{d-1}} h(K,u)^d \, \sigma(\mathrm{d}u).$$

Hölder's inequality and the Aleksandrov–Fenchel inequality give

$$\Phi(K) \geq \alpha V_k(K)^{d/k}$$

with explicit α. Equality holds if and only if K is a centered ball. Hence, the class of centered balls is the limit shape of the typical cell with respect to V_k.

Stability: $\vartheta(K) \geq \epsilon \;\Rightarrow\; \Phi(K) \geq (1 + c_d \epsilon^{(d+3)/2}) \alpha V_k(K)^{d/k}$

(3) The zero cell of a stationary non-isotropic Poisson hyperplane tessellation; the size measured by the inradius

The limit shape of Z_0 with respect to the inradius is the homothety class of the convex body

$$B_\varphi := \bigcap_{u \in \operatorname{supp} \varphi} H(u,1)^-.$$

(4) The zero cell of a stationary isotropic Poisson hyperplane tessellation; the size measured by the circumradius

The limit shape is the class of segments.

(5) The zero cell of a stationary isotropic Poisson hyperplane tessellation; the size measured by the diameter

The limit shape is the class of segments.

(6) The zero cell of a stationary isotropic Poisson hyperplane tessellation; the size measured by the thickness

There is no limit shape, but the class of bodies of constant width can be considered as the asymptotic shape.

(7) The zero cell of a stationary isotropic Poisson hyperplane tessellation; the size measured by the width in a given direction

The limit shape is the class of segments of the given direction.

An unsolved problem:

Replace size by large number of facets or vertices.

Poisson–Delaunay Mosaics

A Poisson–Delaunay mosaic is the dual tessellation of a stationary Poisson–Voronoi tessellation. The cells are simplices.

What is the asymptotic shape of 'large' typical cells?

Results

The shape of the typical cell tends to the shape of a regular simplex, given that the volume, or the surface area, or the inradius, or the minimal width, of the typical cell tends to infinity.

Typical cells of large diameter tend to belong to a special class of simplices, distinct from the regular ones. In the plane, these are the right-angled triangles.

The Size Functional

A d-simplex S has a circumscribed sphere (through the vertices); let $z(S)$ be its centre and $R(S)$ its radius. Let Δ_0 be the space of d-simplices $S \subset \mathbb{R}^d$ with $z(S) = 0$.

Let $\Sigma : \Delta_0 \to \mathbb{R}$ be a function with the following properties:

- continuous,
- homogeneous of some degree $k > 0$,
- Σ attains a maximum on the set of d-simplices inscribed to the unit sphere,
- $V_d/\Sigma^{1/k}$ is bounded.

We call Σ the size functional.

Again, a crucial inequality:

$$\Sigma(S) \leq \alpha R(S)^k \qquad \forall\, S \in \Delta_0,$$

with equality attained by the extremal simplices.

The deviation functional: a function $\vartheta : \Delta_0 \to \mathbb{R}$ with the properties

- continuous,
- homogeneous of degree 0,
- $\vartheta(S) = 0 \ \Leftrightarrow\ S$ is an extremal simplex.

The stability function: $f : [0,1] \to [0,1]$ with $f(0) = 0$, $f(\epsilon) > 0$ for $\epsilon > 0$, and

$$\vartheta(S) \geq \epsilon \ \Rightarrow\ \Sigma(S) \leq (1 - f(\epsilon))\alpha R(S)^k.$$

Theorem 3. *Let Z be the typical cell of the Poisson–Delaunay tessellation derived from a stationary Poisson point process with intensity $\gamma > 0$ in $\mathbb{R}^d$.*

If $\epsilon \in (0,1)$ and $a > 0$, then

$$\mathbb{P}(\vartheta(Z) \geq \epsilon \mid \Sigma(Z) \geq a) \leq c \exp\left\{-c_0 f(\epsilon) a^{d/k} \gamma\right\},$$

where c, c_0 are constants depending on d, ϑ, f, and c also on ϵ.

Unsolved problems

1.) The simplices of maximal surface area contained in a ball are precisely the regular ones. Prove a stability result.

2.) Which simplices contained in a ball have maximal mean width?