Journées de Géométrie Algorithmique, January 2009

Proximity of Persistence Modules and their Diagrams

Steve Oudot

→ joint work with F. Chazal, D. Cohen-Steiner, M. Glisse, L. J. Guibas

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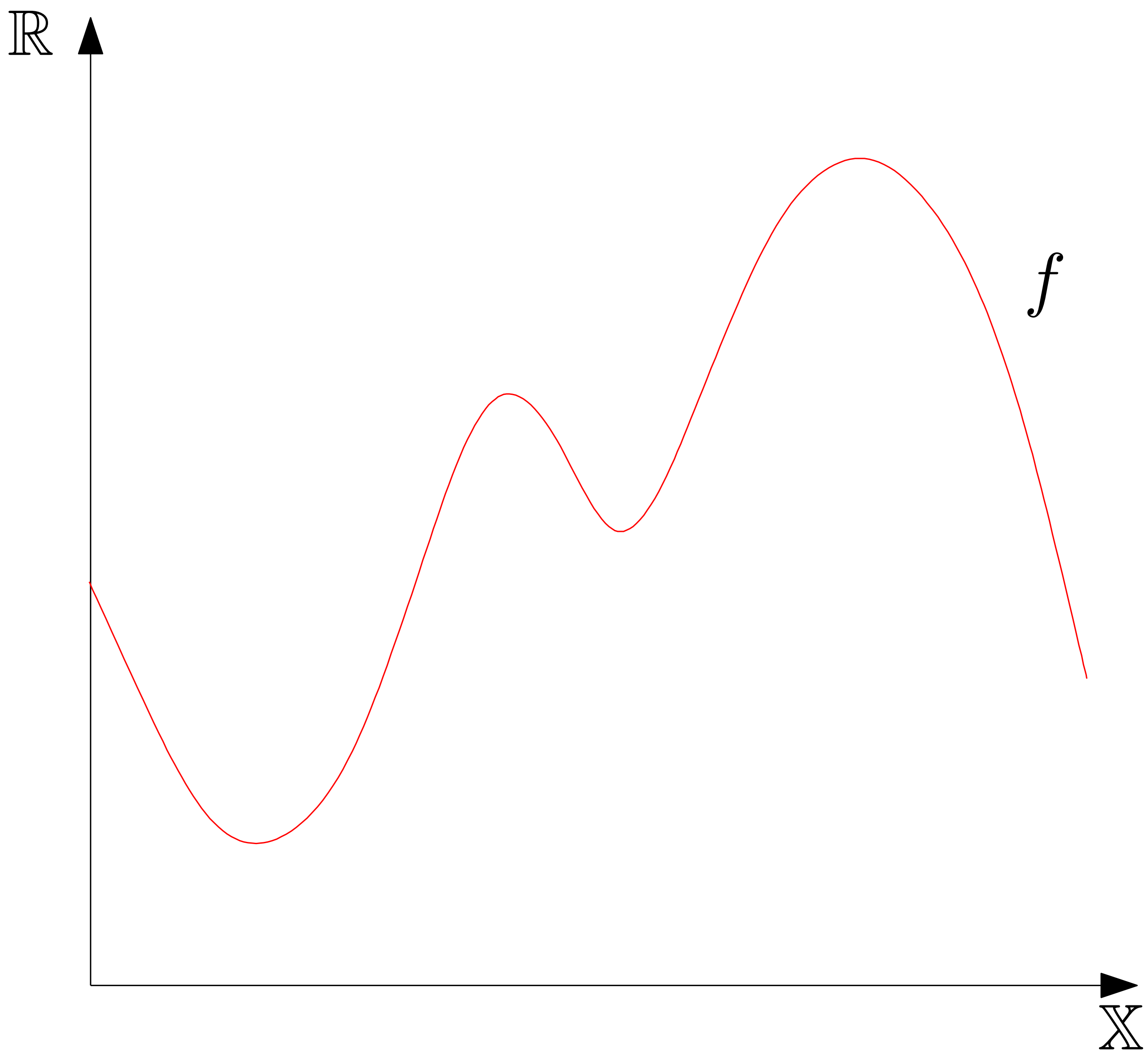
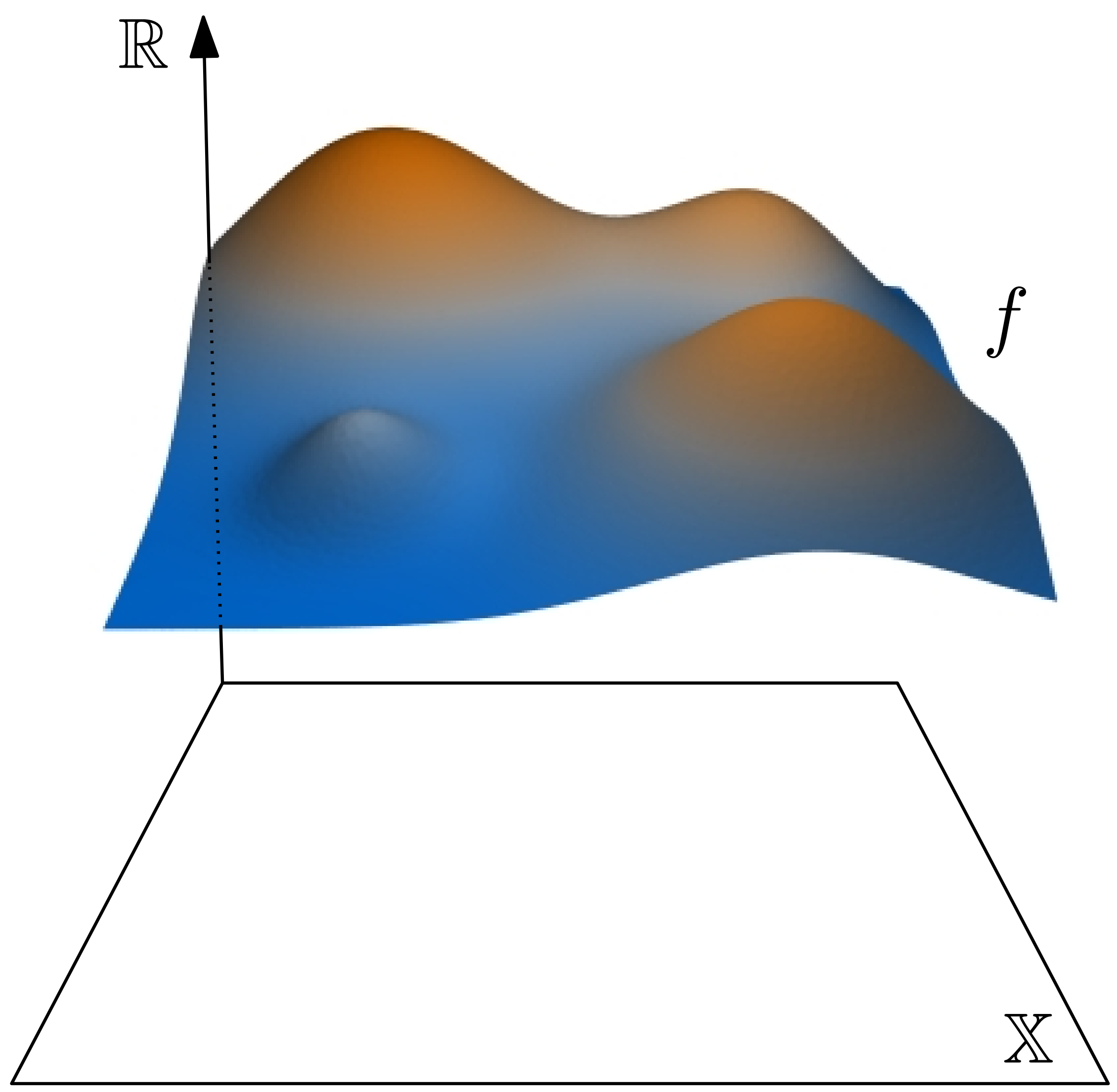
Topological Persistence (in a nutshell)

$\mathbb{X}$ topological space, $f : \mathbb{X} \to \mathbb{R}$ function, $k \in \mathbb{Z}$.





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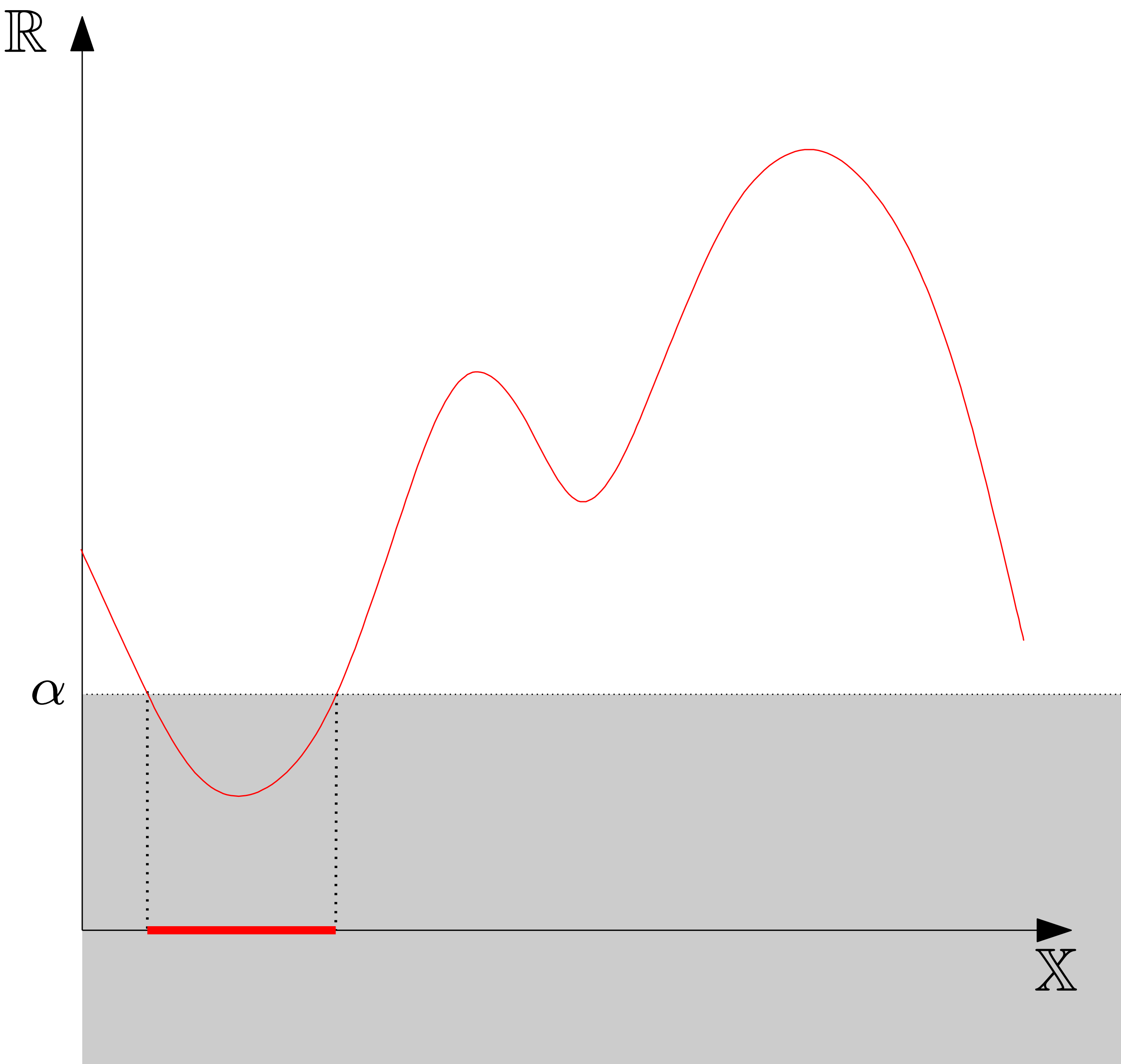

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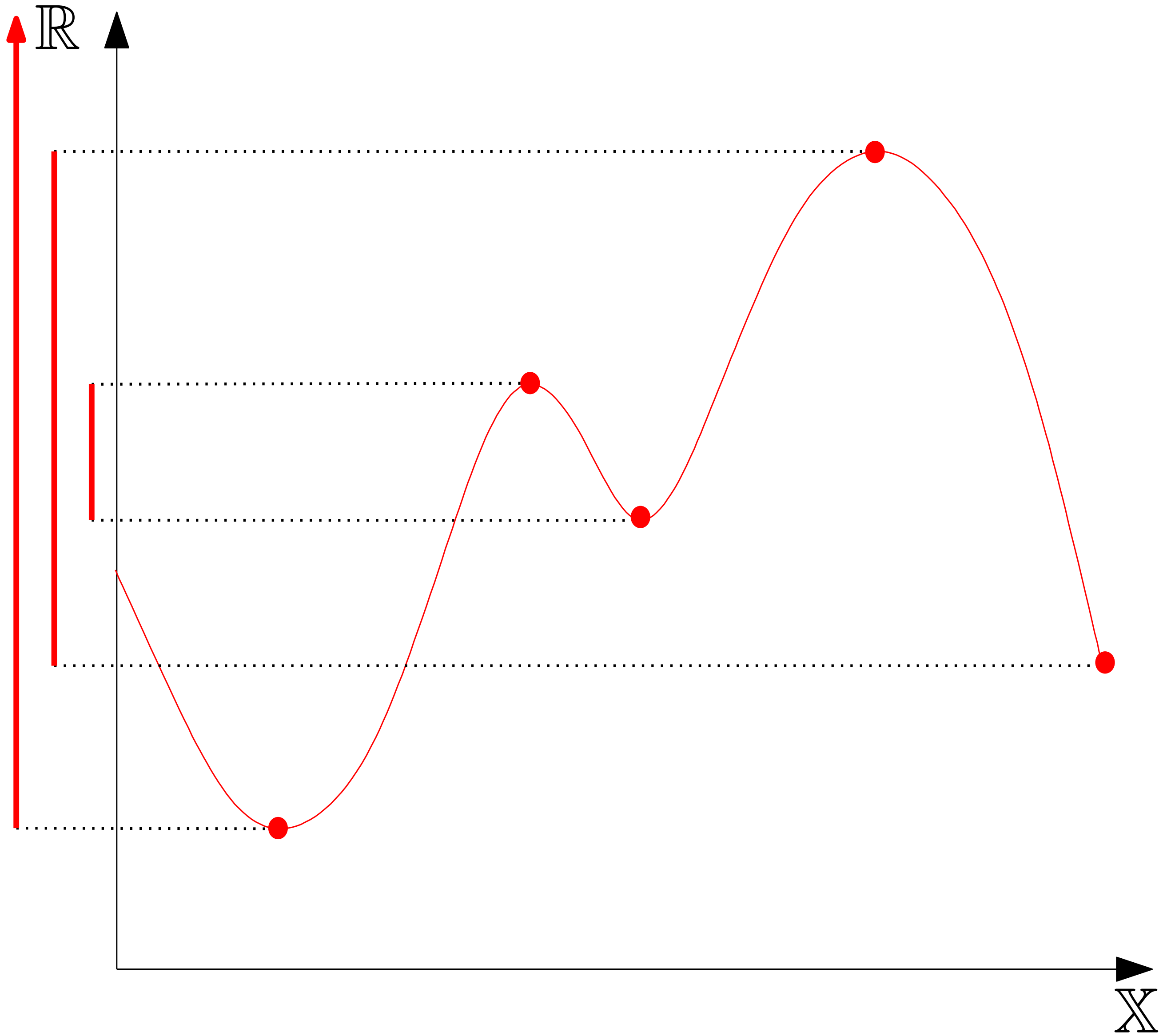

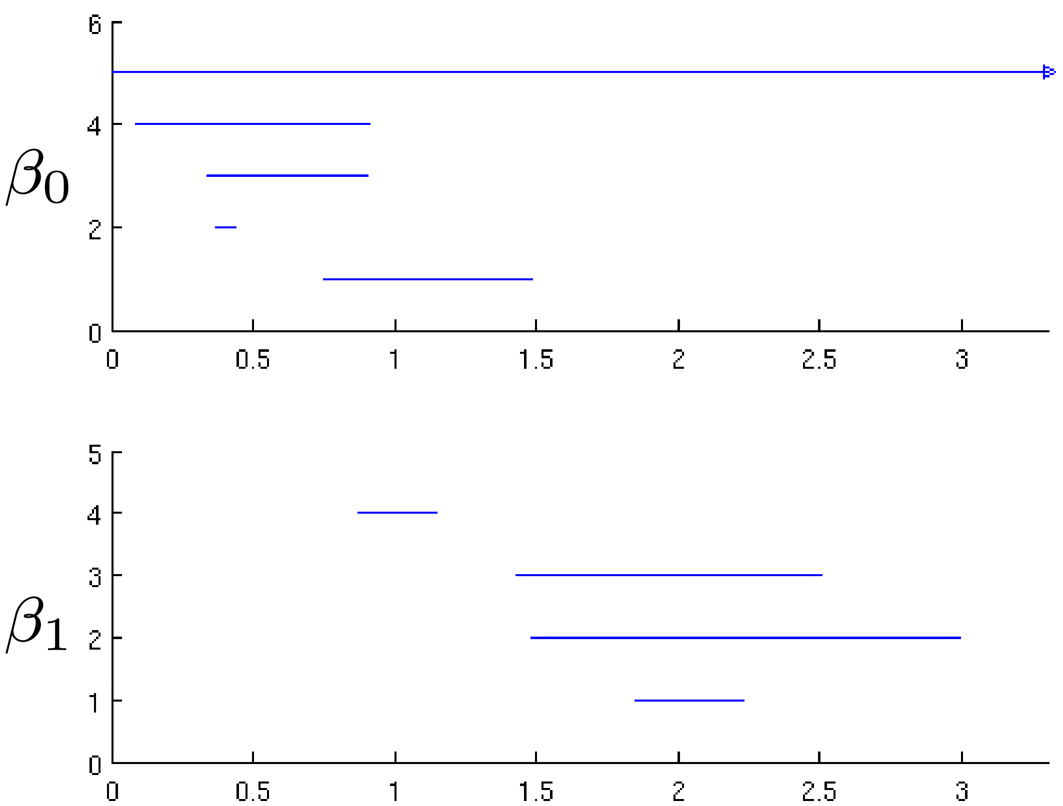

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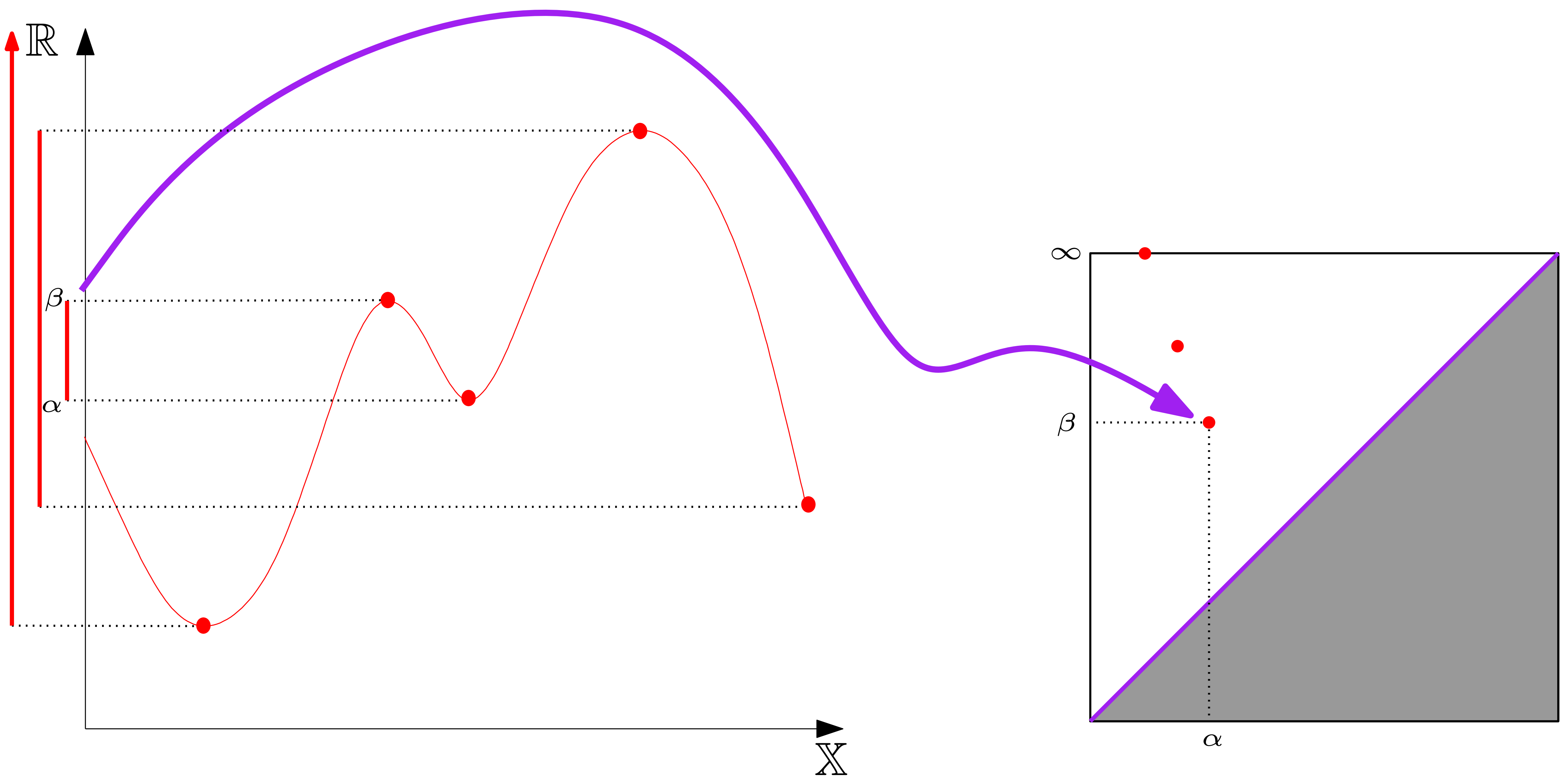

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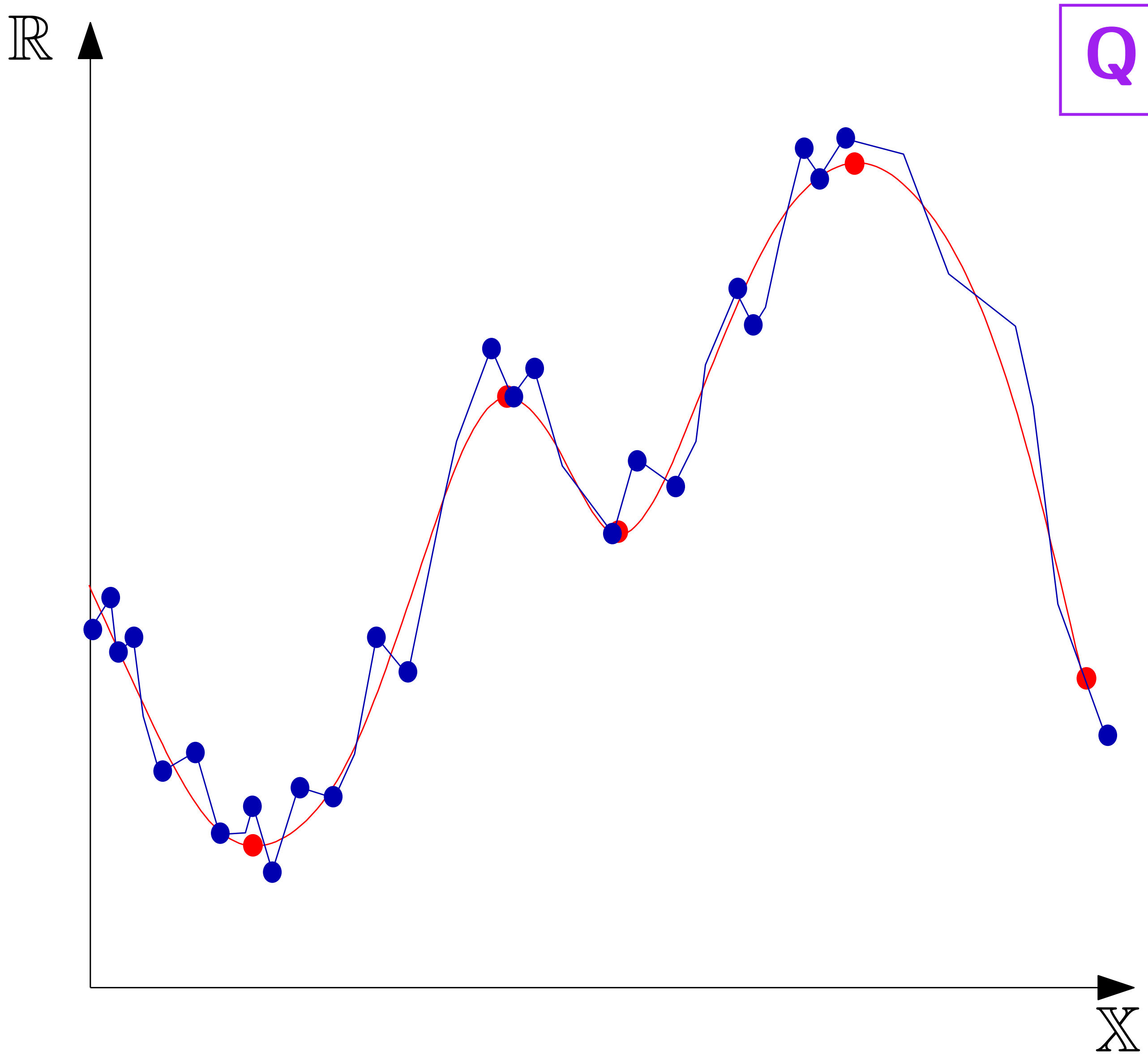
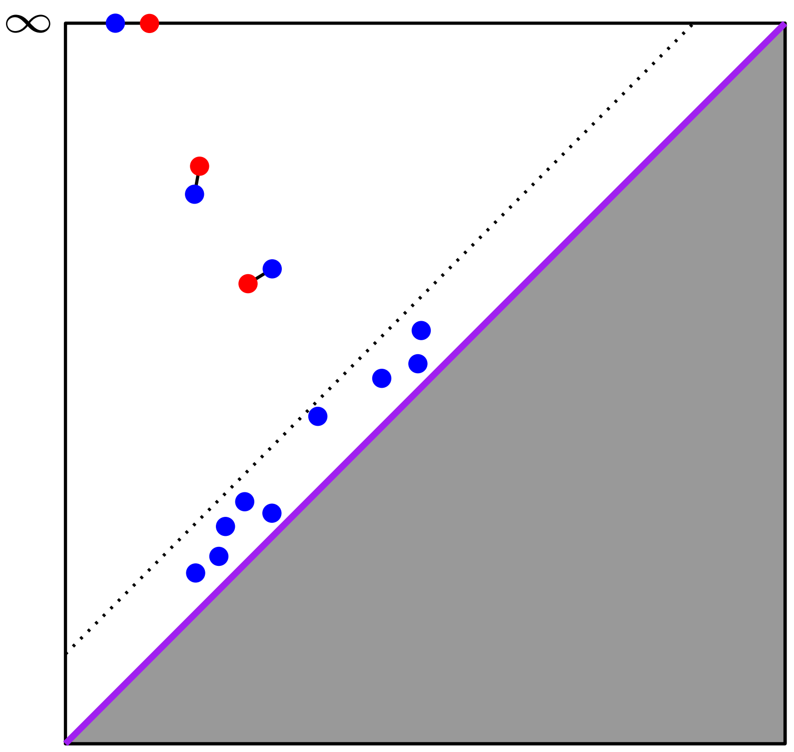
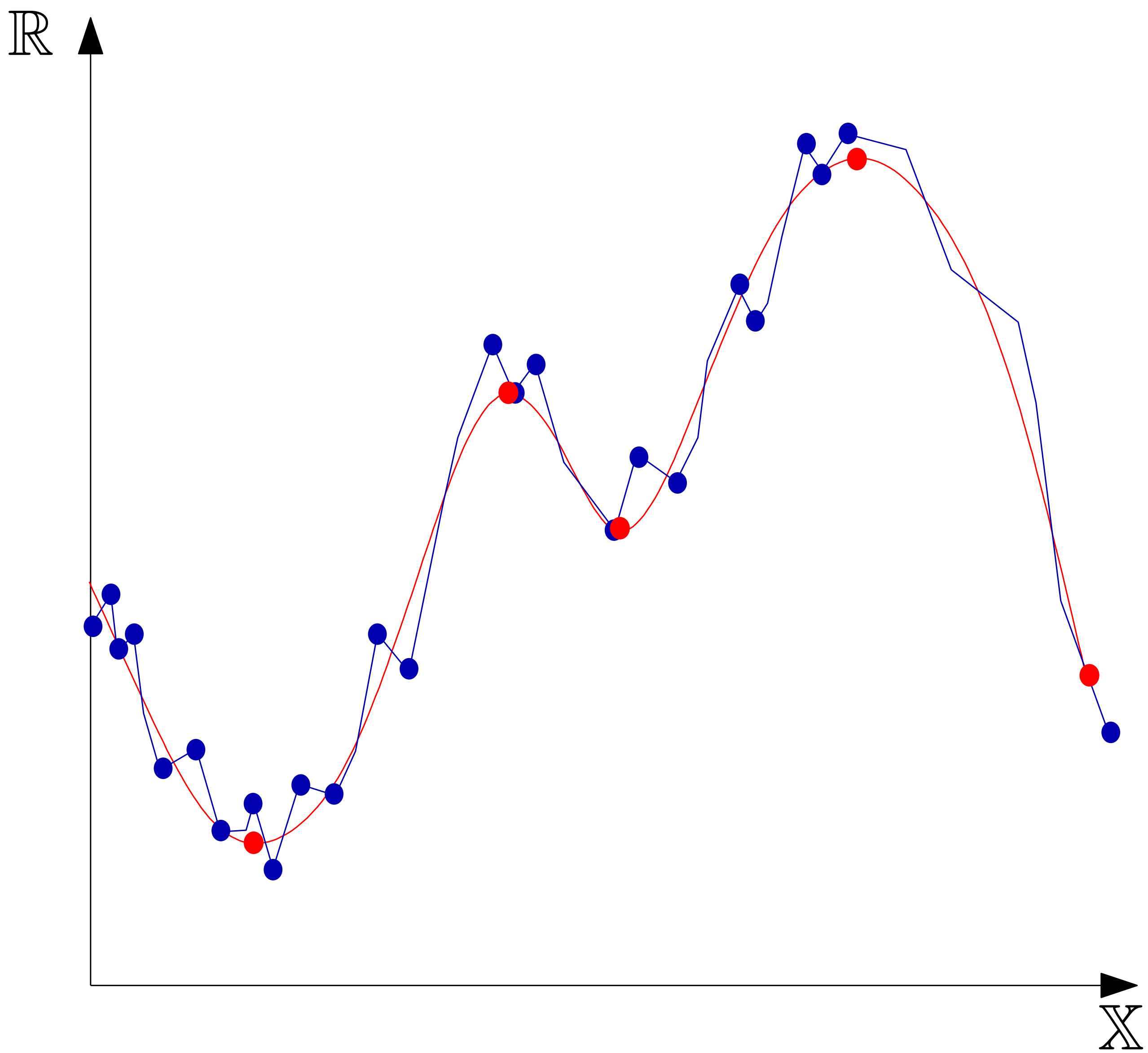
Q What if f is slightly perturbed?

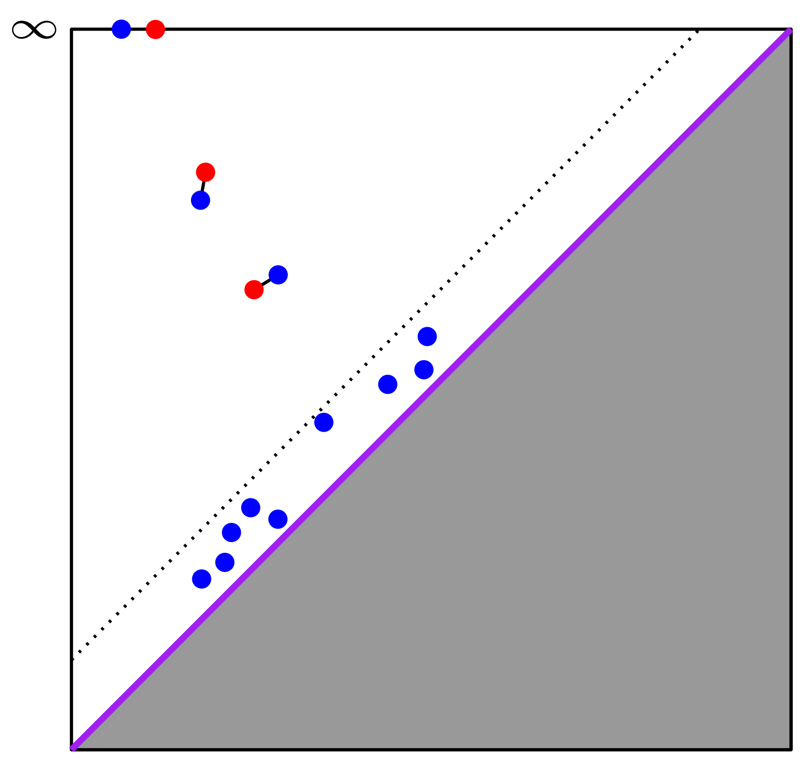




Perturbations and Stability Results

Theorem: [Cohen-Steiner, Edelsbrunner, Harer 05]
Let $f, g : \mathbb{X} \to \mathbb{R}$ be two tame functions. If f, g are continuous and $\mathbb{X}$ is triangulable, then, $\forall k \in \mathbb{Z}$, $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\ f, \mathrm{D}_k\ g) \leq \|f - g\|_{\infty}$.





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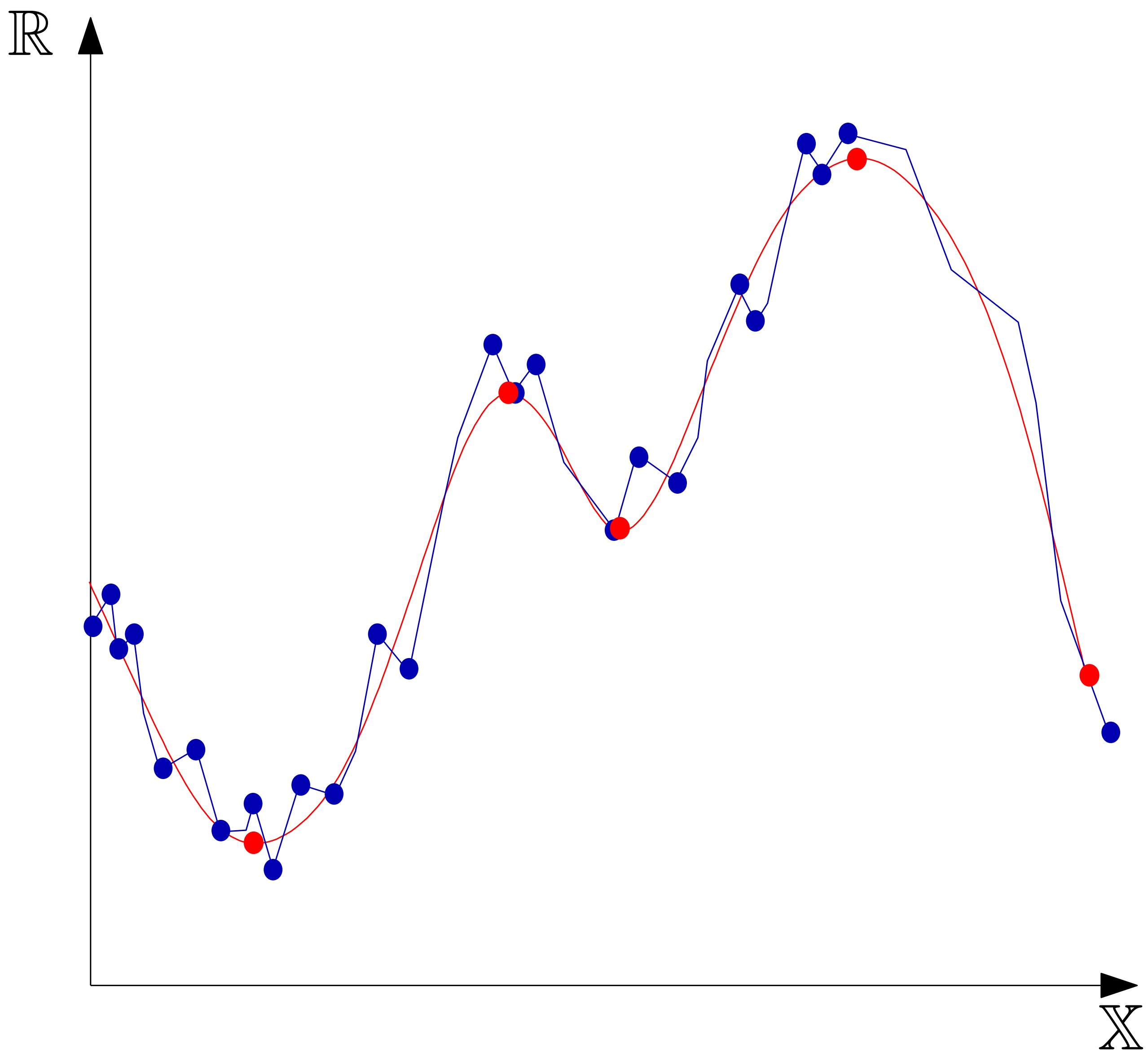
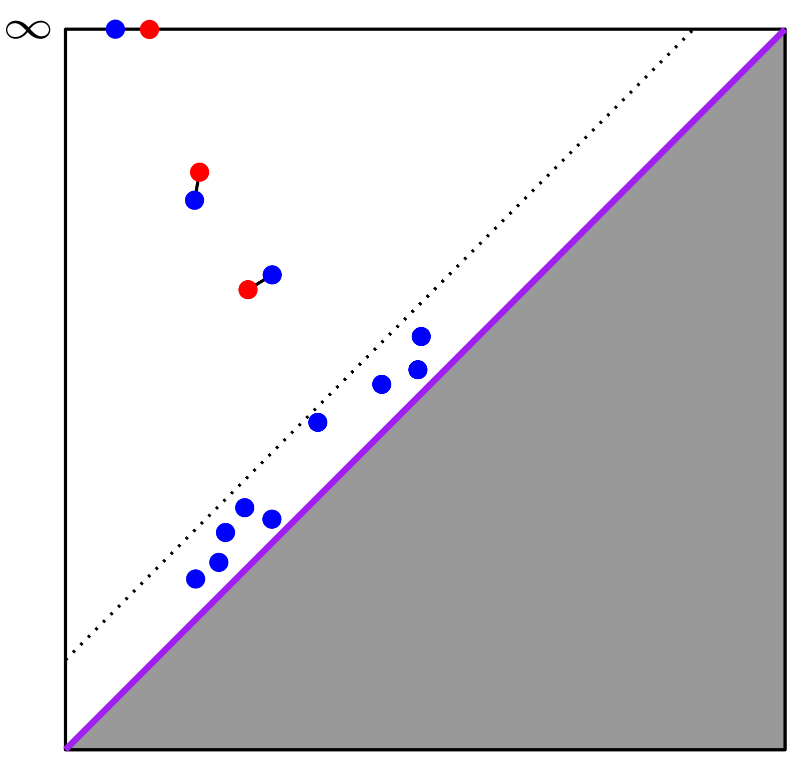
Can these conditions be removed?





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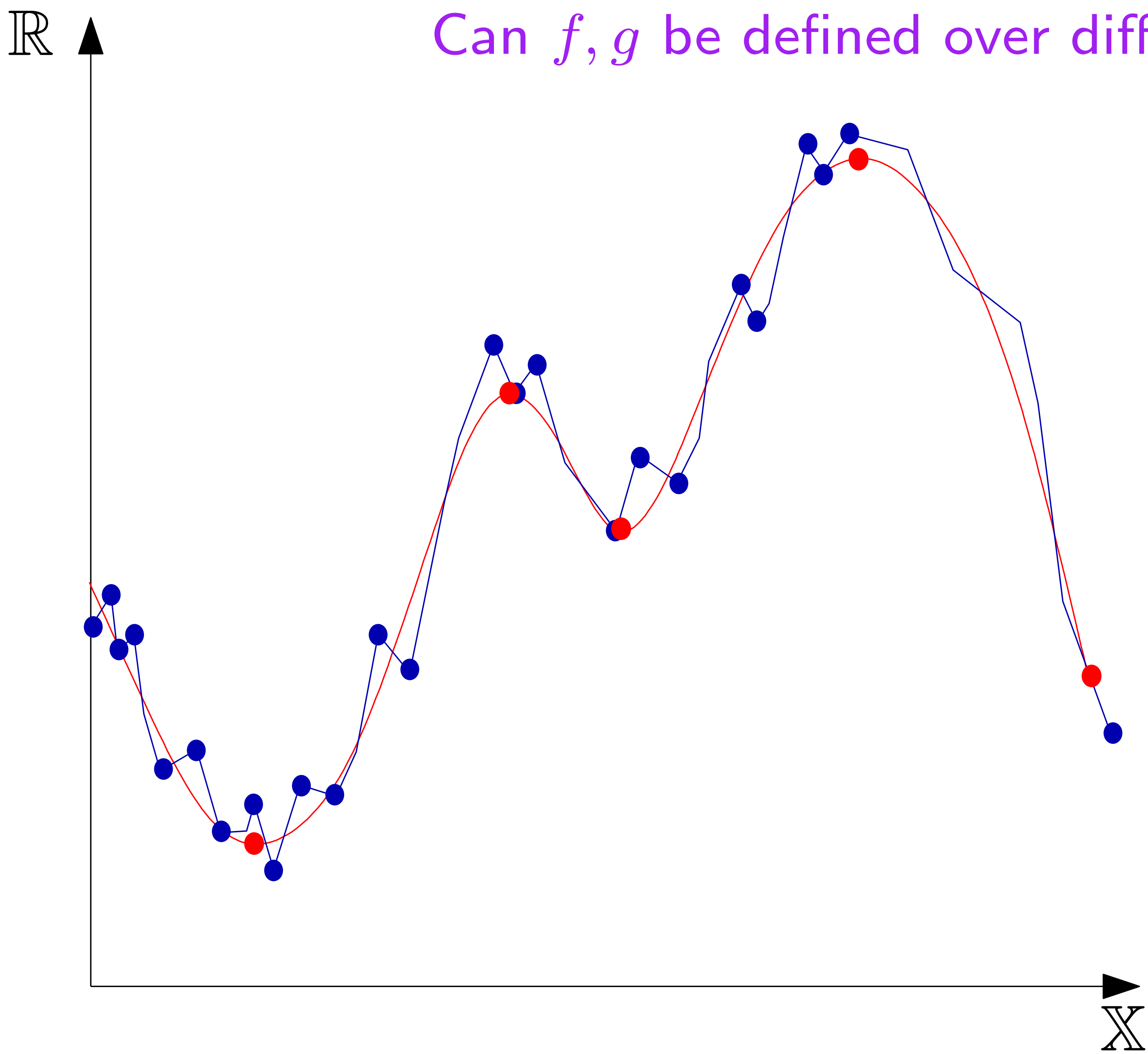
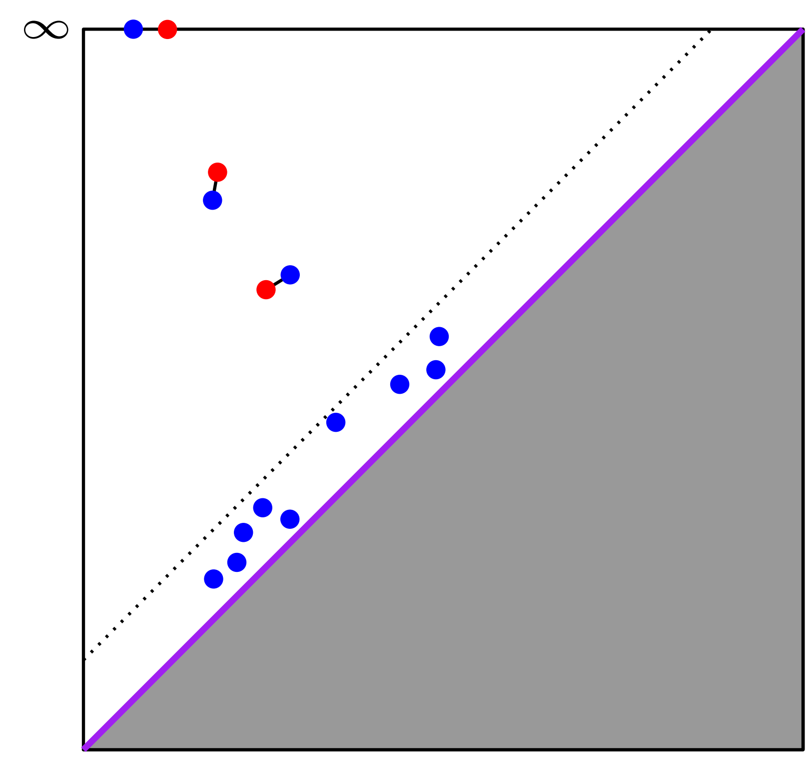
Can these conditions be removed?

Can f, g be defined over different spaces $\mathbb{X}, \mathbb{Y}$?





Outline

- Overview of the proof of [CEH'05] — where continuity and triangulability are needed;

- A new, simple, geometrically-flavored proof of stability with an upper bound of $3\|f - g\|_\infty$ on the bottleneck distance;

- Reducing the upper bound from $3\|f - g\|_\infty$ to $\|f - g\|_\infty$ — interpolating at algebraic level directly.

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Proof Overview

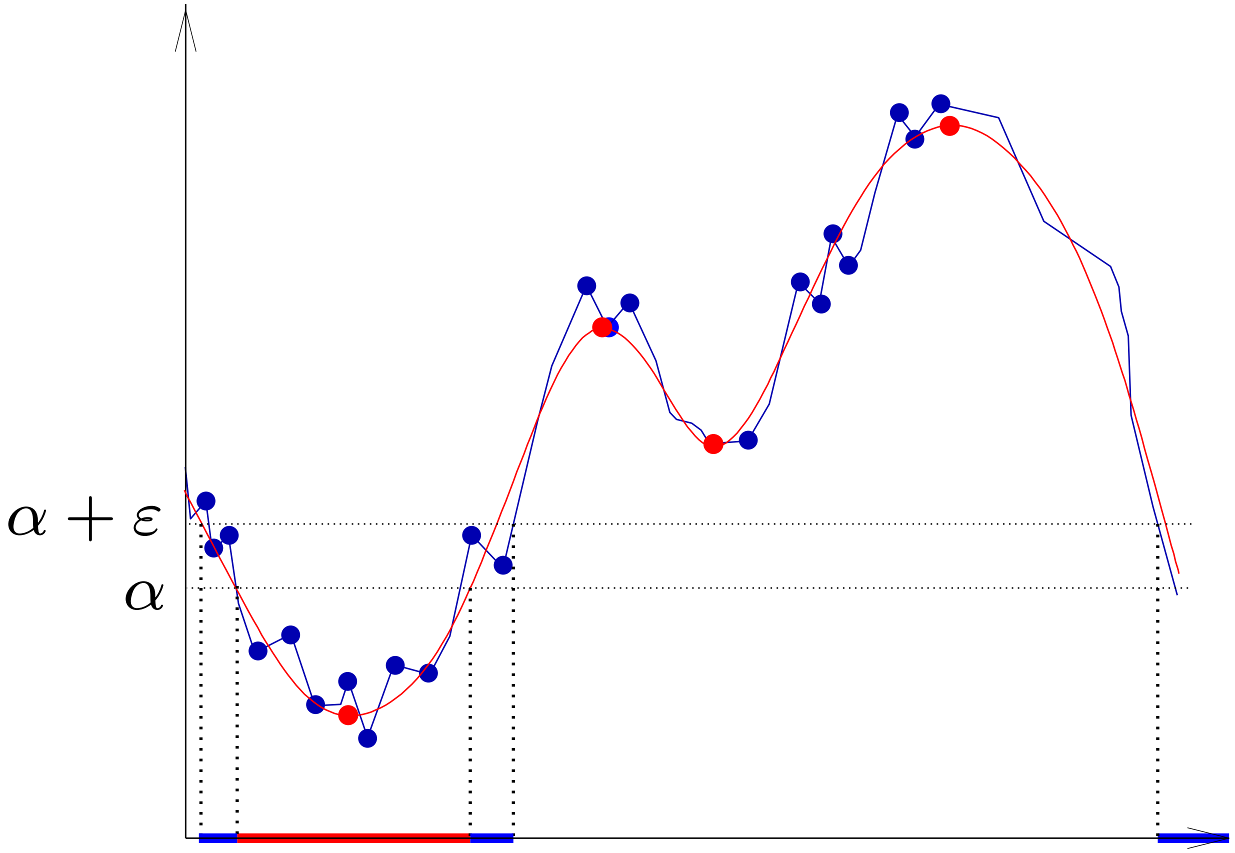
$F_\alpha := f^{-1}((-\infty, \alpha])$

$G_\alpha := g^{-1}((-\infty, \alpha])$

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

- Key observation: $\{F_\alpha\}_\alpha$ and $\{G_\alpha\}_\alpha$ are ε-**interleaved** w.r.t. inclusion:

$$\forall \alpha \in \mathbb{R},\ F_{\alpha-\varepsilon} \subseteq G_\alpha \subseteq F_{\alpha+\varepsilon}.$$



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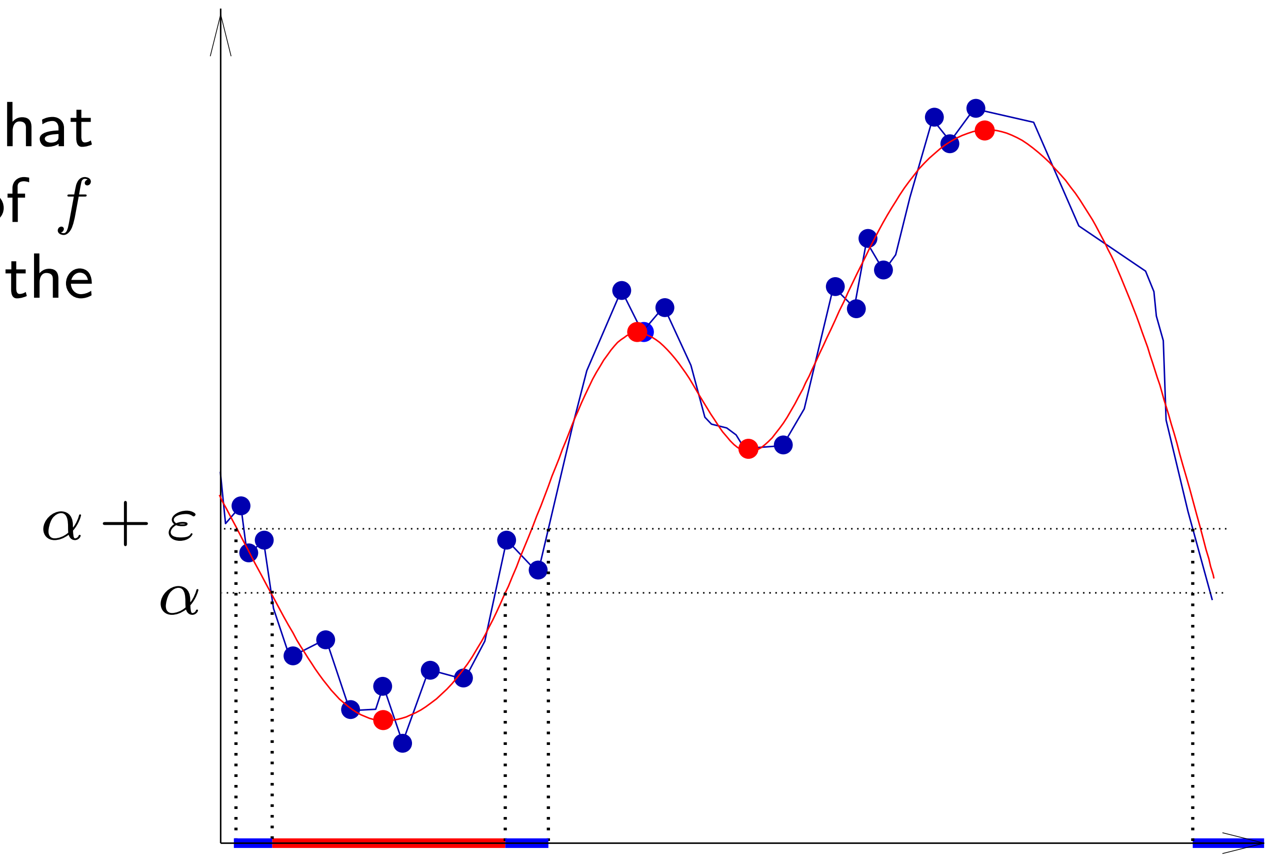
$\to$ Intuition: every homological feature that appears/dies at time α in the filtration of f appears/dies at time $\alpha + \varepsilon$ at the latest in the filtration of g, and vice versa.



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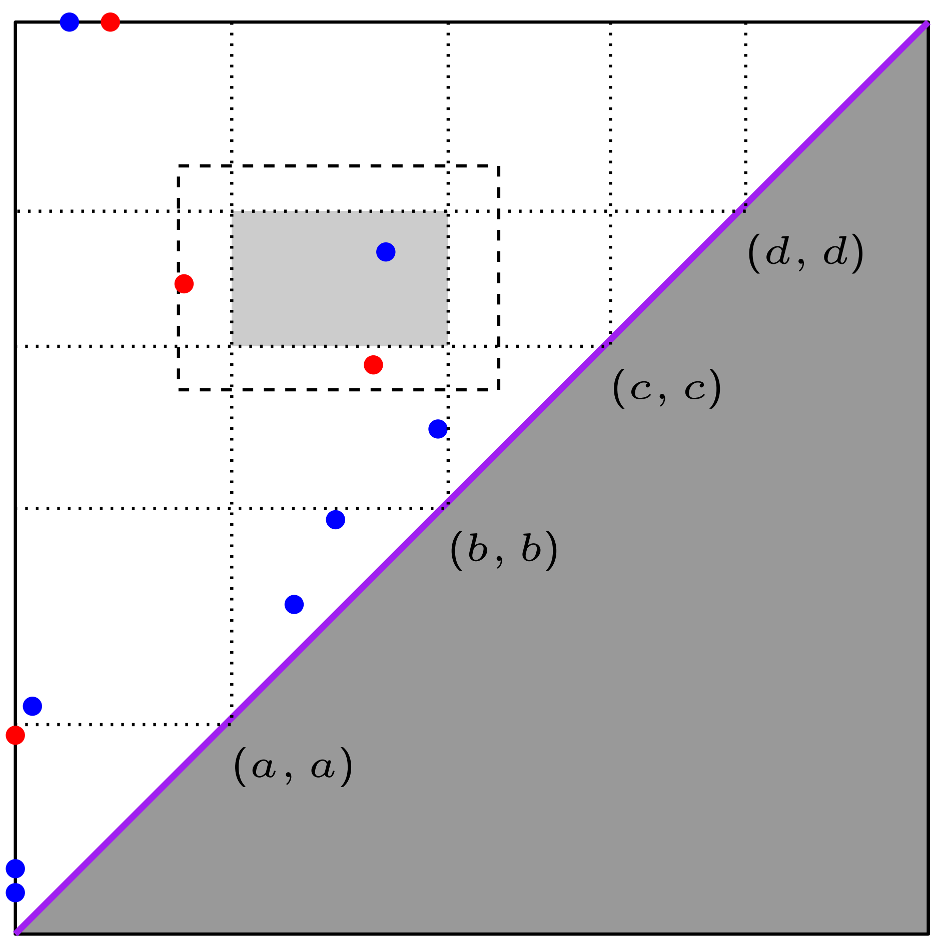
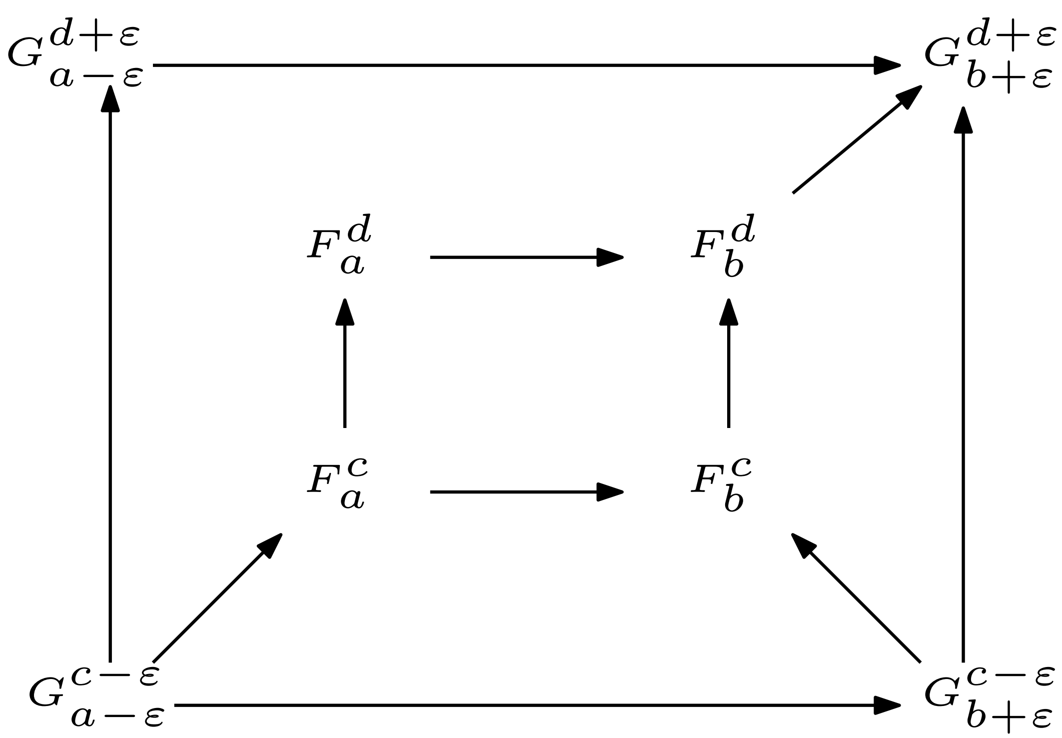
- Bound on Hausdorff distance: $\mathrm{d}_\mathrm{H}^\infty(\mathrm{D}_k\, f, \mathrm{D}_k\, g) \leq \varepsilon$

$\rightarrow$ Box Lemma: $\forall$ box $\square$, $\#(\mathrm{D}_k\, f \cap \square) \leq \#(\mathrm{D}_k\, g \cap \square_\varepsilon)$





$F_\alpha^\beta = \mathrm{im}\ H_k(F_\alpha) \to H_k(F_\beta)$

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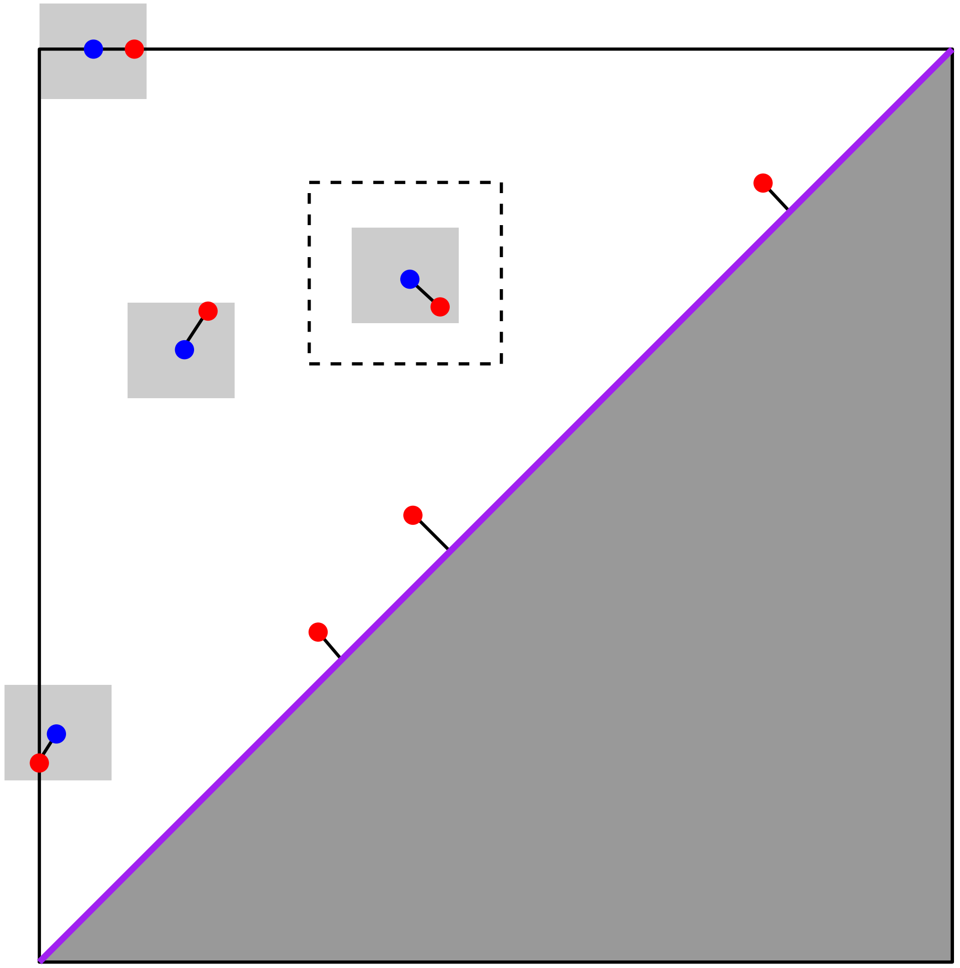
- From Hausdorff to bottleneck: the *infinitesimal* case:



Let $\delta_f = \min\{|a-b|,$ a,b hcv of f$\}$

Assume that $\varepsilon < \frac{\delta_f}{4}$

Box Lemma $\Rightarrow \forall p \in \mathrm{D}_k\, f \setminus \Delta$,
$\mu(\{p\}) = \#(\mathrm{D}_k\, g \cap \{p\}_\varepsilon) = \#(\mathrm{D}_k\, f \cap \{p\}_{2\varepsilon})$

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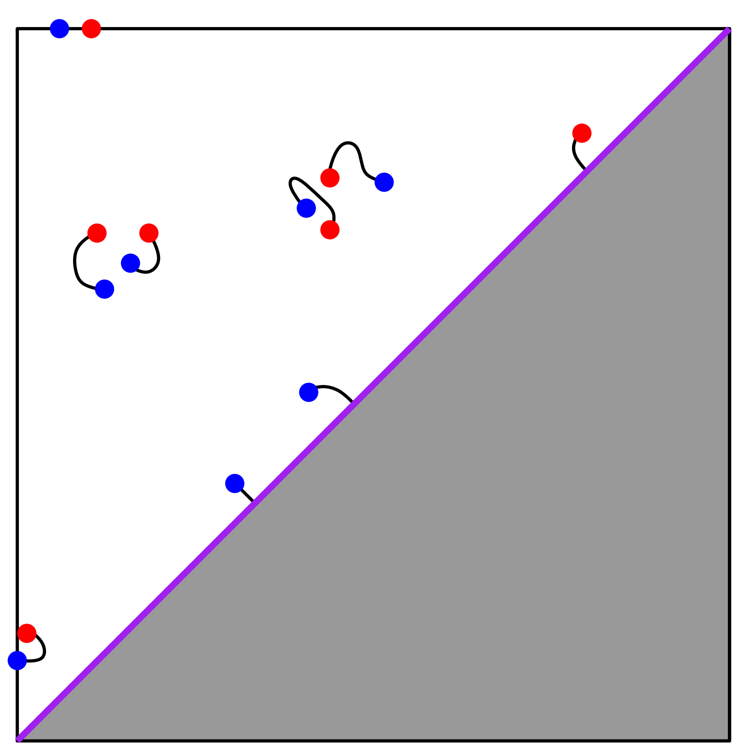

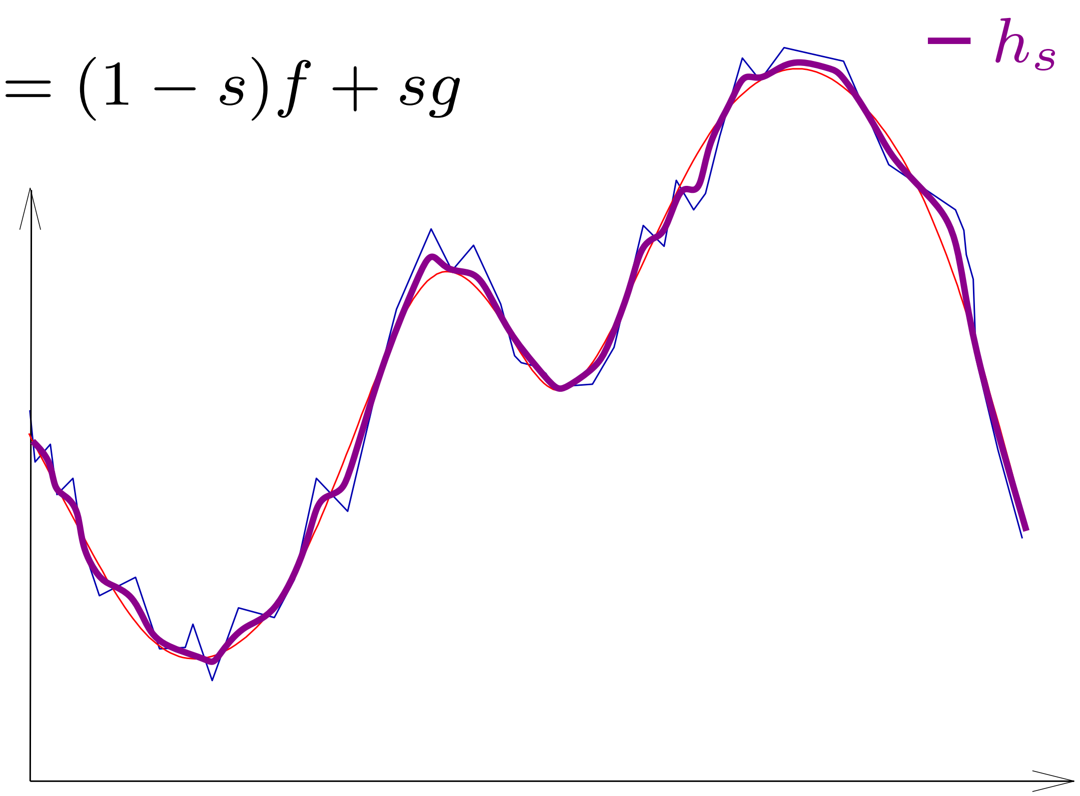

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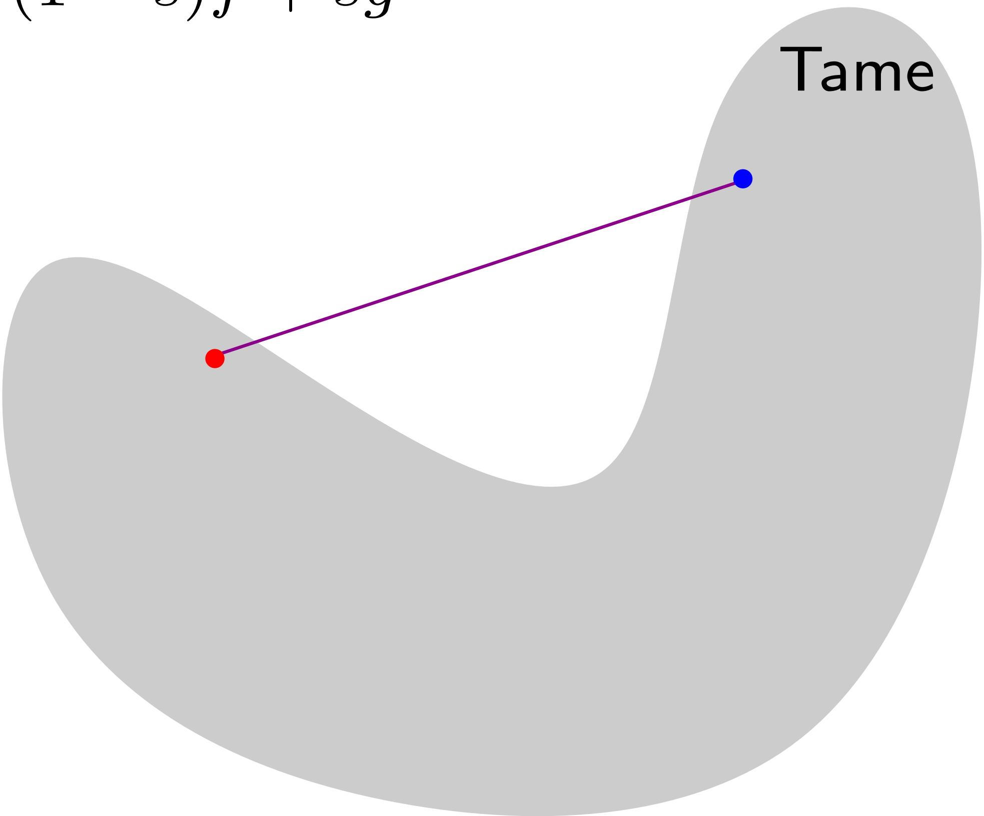

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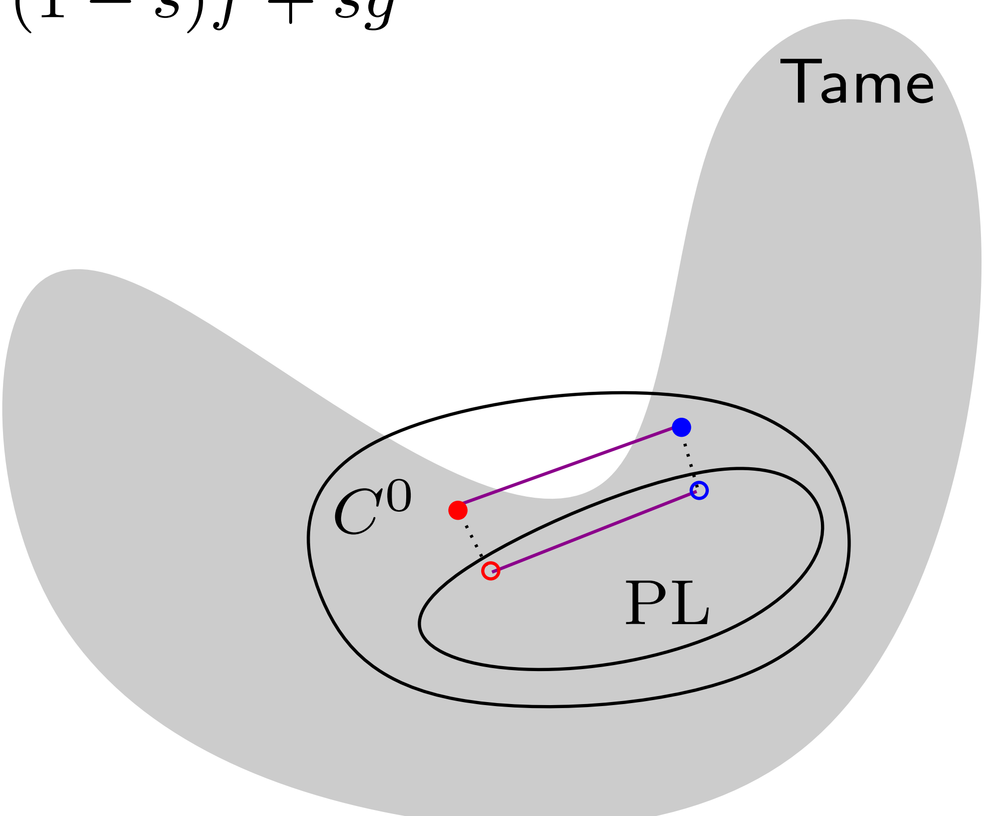
$\to$ assume $\mathbb{X}$ is triangulable and f, g are C^0

$\mathrm{PL}(\mathbb{X})$ is convex, in Tame($\mathbb{X}$), and dense in $C^0(\mathbb{X})$

$\to$ build PL interpolations $\hat{f}, \hat{g}$ of f, g

$\to$ interpolate between $\hat{f}$ and $\hat{g}$



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- both filtrations are 2ε-pixelizations of $\{H_{n\varepsilon}\}_{n\in\mathbb{Z}}$, where $H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases}$

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$\longrightarrow$ **goal**: bound distances between diagrams of functions and pixelizations

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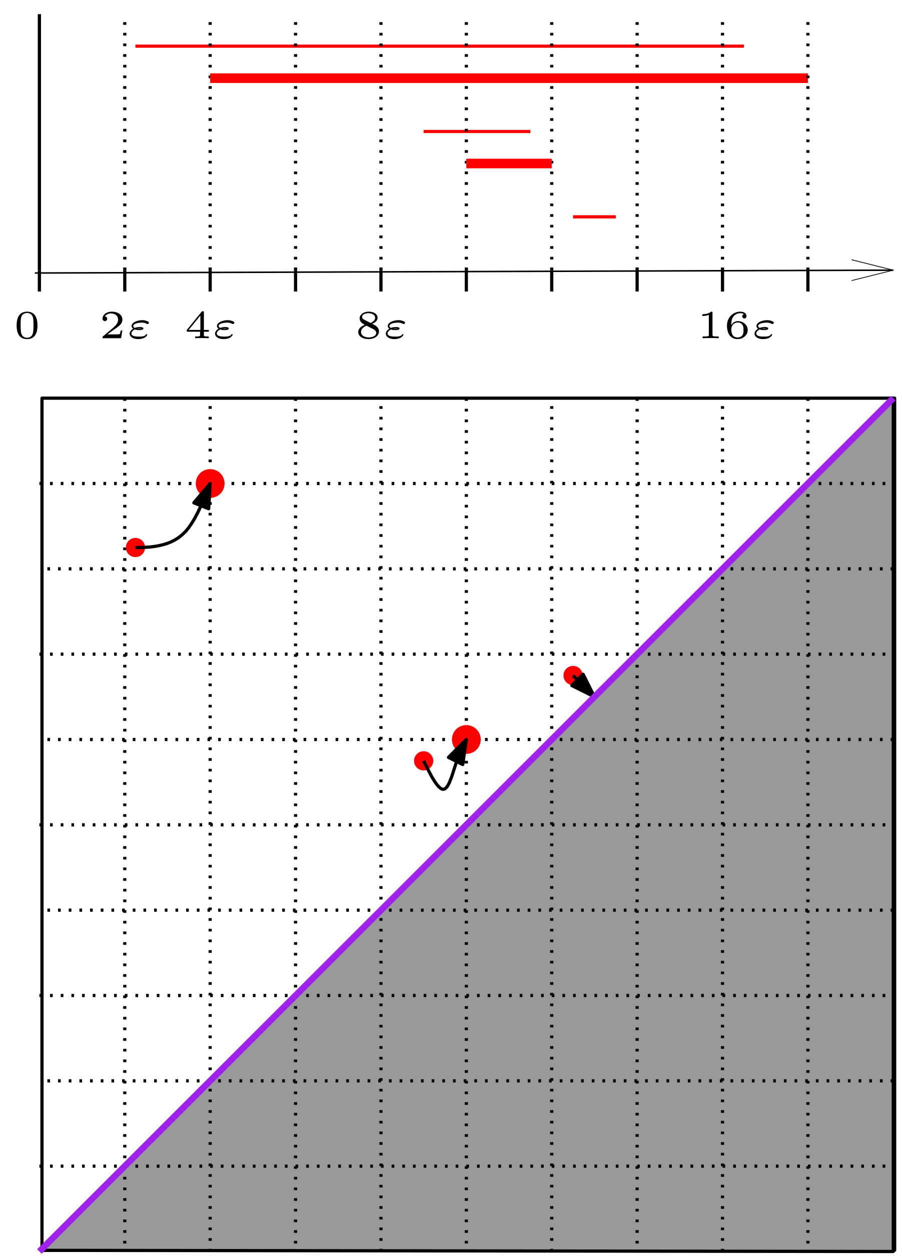
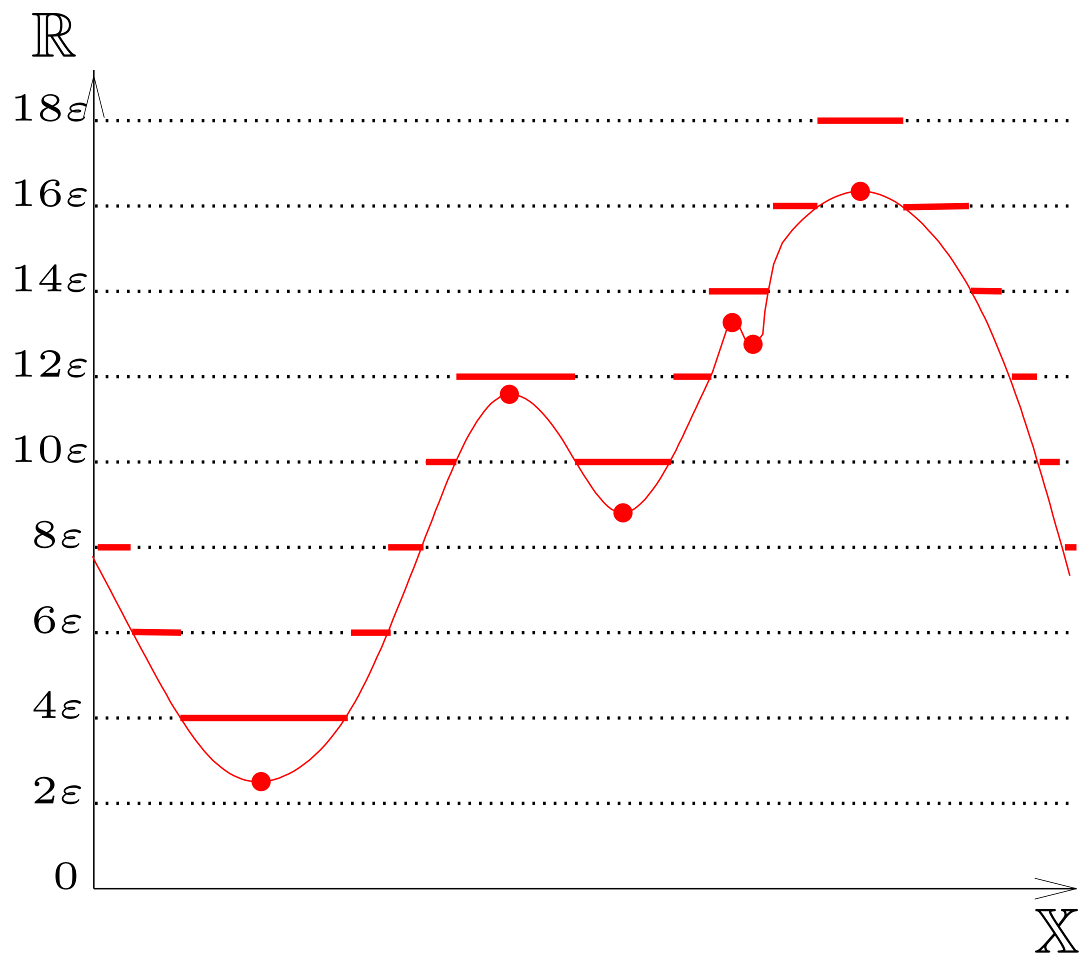
- Pixelizations: $\longrightarrow$ effect on persistence barcode/diagram:





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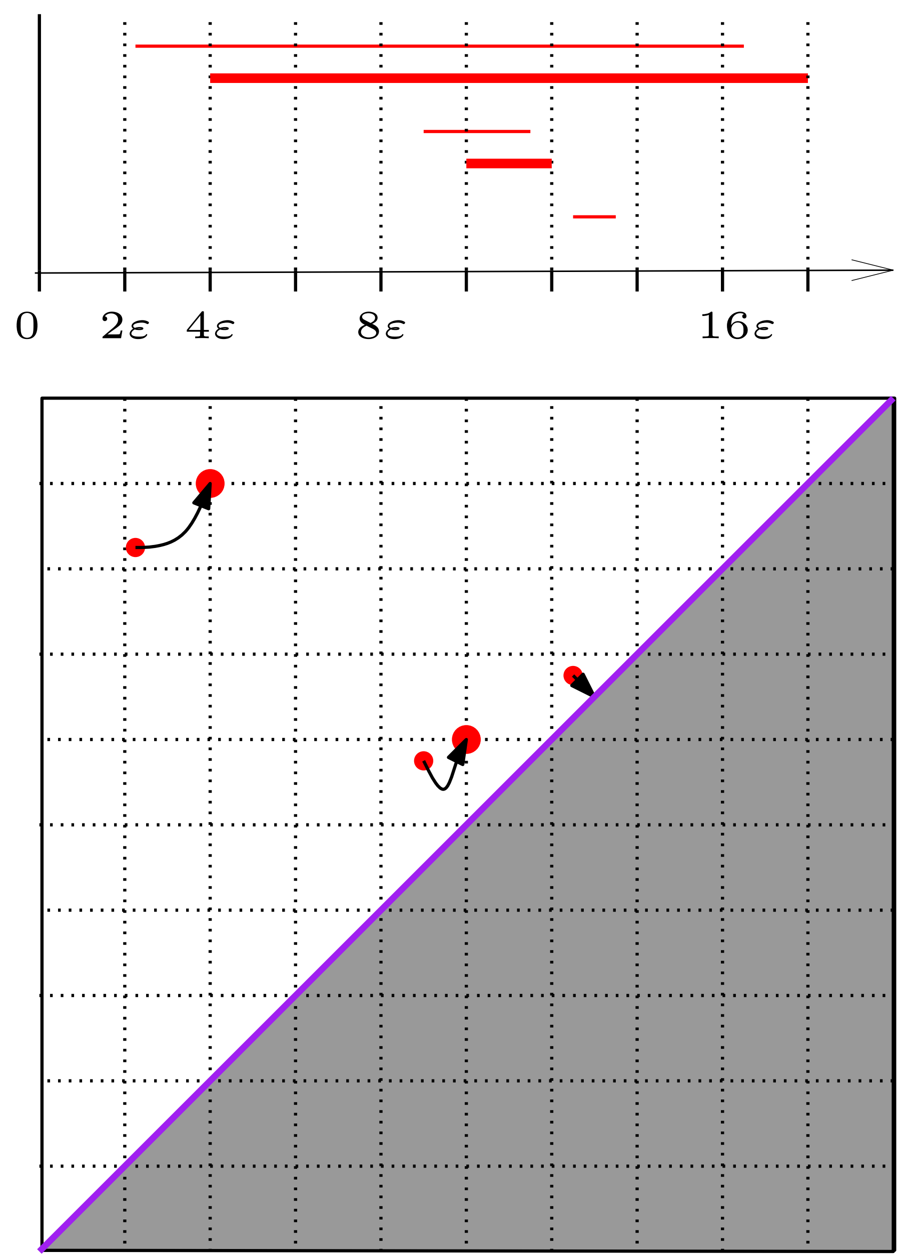

Pixelization map: $\forall \alpha \leq \beta$,

$$\pi_{2\varepsilon}(\alpha, \beta) = \begin{cases} (\lceil \frac{\alpha}{2\varepsilon} \rceil 2\varepsilon, \lceil \frac{\beta}{2\varepsilon} \rceil 2\varepsilon) \text{ if } \lceil \frac{\beta}{2\varepsilon} \rceil > \lceil \frac{\alpha}{2\varepsilon} \rceil \\ (\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}) \text{ if } \lceil \frac{\beta}{2\varepsilon} \rceil = \lceil \frac{\alpha}{2\varepsilon} \rceil \end{cases}$$

Theorem: If $f : \mathbb{X} \to \mathbb{R}$ is tame, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{D}_k\, f \to \mathrm{D}_k\, f^{2\varepsilon}$.

$$\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\, f, \mathrm{D}_k\, f^{2\varepsilon}) \leq 2\varepsilon$$

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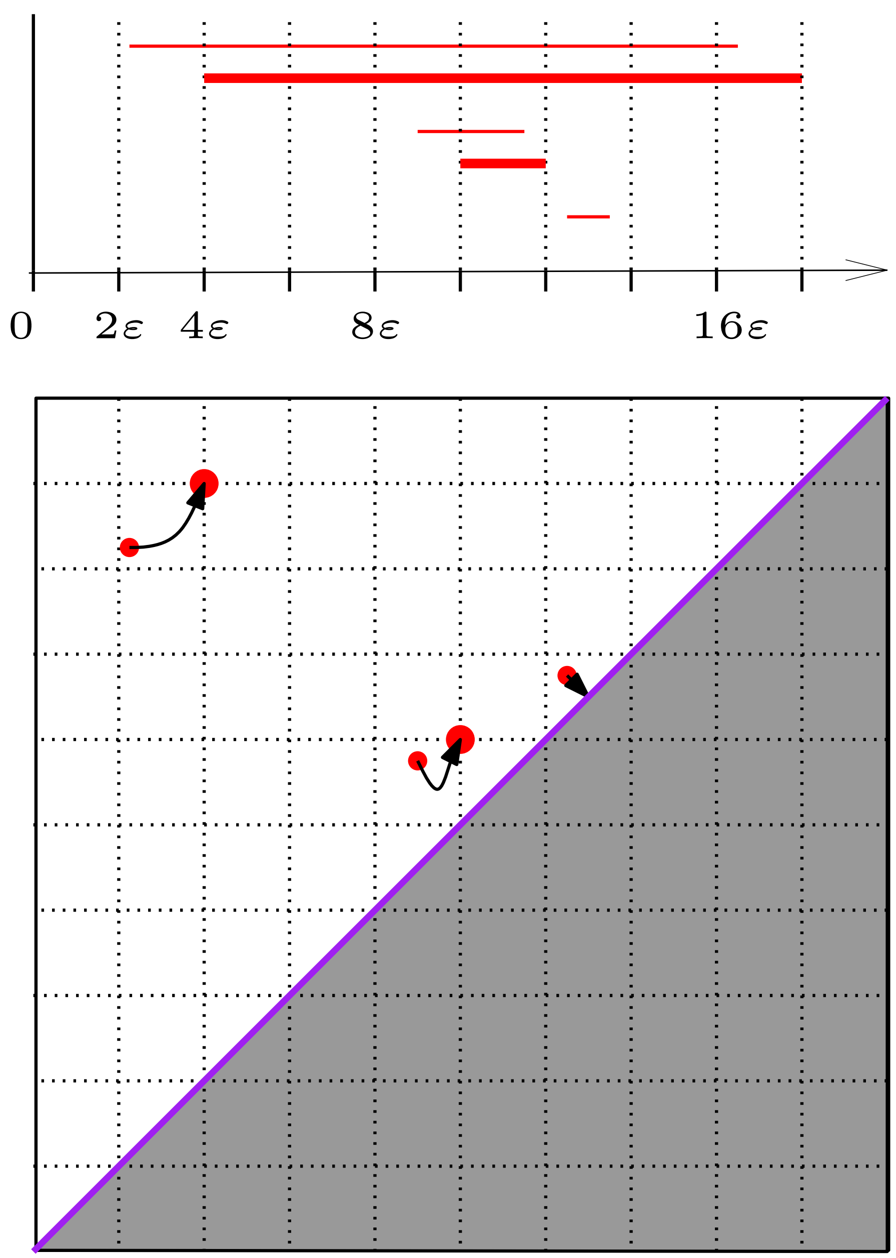

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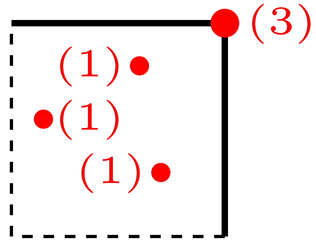
Theorem: If $f : \mathbb{X} \to \mathbb{R}$ is tame, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{D}_k\ f \to \mathrm{D}_k\ f^{2\varepsilon}$.

$\longrightarrow$ proof: show that the multiplicities of $\mathrm{D}_k\ f$ and $\mathrm{D}_k\ f^{2\varepsilon}$ are the same inside each grid cell that does not intersect the diagonal.

The case of diagonal cells is trivial.



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Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

- Back to interleaved filtrations:

$$F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq F_{(2n+2)\varepsilon} \subseteq \cdots$$

$$H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases} \quad \to \left(\begin{array}{l} F^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } F \text{ and } H \\ G^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } G \text{ and } H \textbf{ of phase } \varepsilon \end{array} \right.$$

Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\, f, \mathrm{D}_k\, g) \leq 8\varepsilon$

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Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

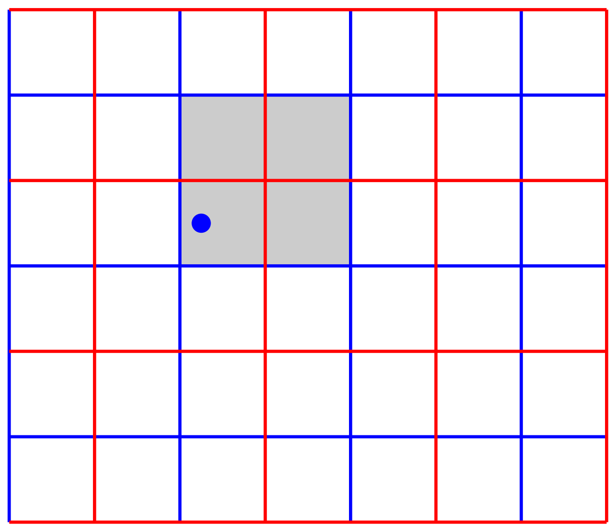
- Back to interleaved filtrations:

$$F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq F_{(2n+2)\varepsilon} \subseteq \cdots$$

$$H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases} \quad \to \left(\begin{array}{l} F^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } F \text{ and } H \\ G^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } G \text{ and } H \textbf{ of phase } \varepsilon \end{array} \right.$$

Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_\mathrm{B}^\infty(\mathrm{D}_k\, f, \mathrm{D}_k\, g) \leq 8\varepsilon$

Improvement:



A Simple Proof

$$F_\alpha := f^{-1}((-\infty, \alpha])$$
$$G_\alpha := g^{-1}((-\infty, \alpha])$$

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

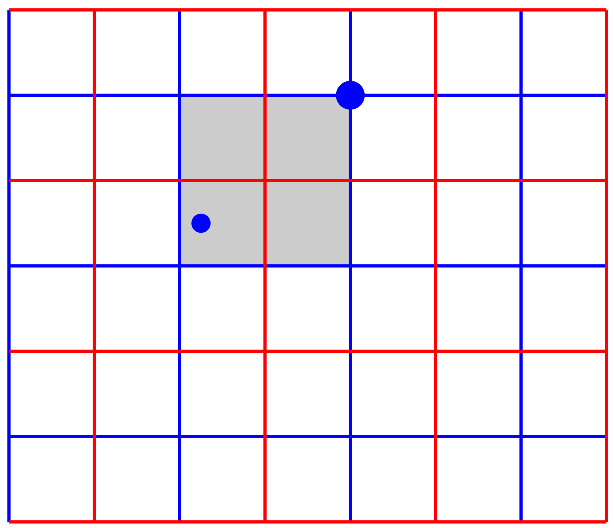
- Back to interleaved filtrations:

$$F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq F_{(2n+2)\varepsilon} \subseteq \cdots$$

$H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases} \quad \to \quad$ $F^{2\varepsilon}$ is the 2ε-pixelization of F and H; $G^{2\varepsilon}$ is the 2ε-pixelization of G and H **of phase** ε

Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\, f, \mathrm{D}_k\, g) \leq 8\varepsilon$

Improvement:



A Simple Proof

$$\left| \begin{array}{l} F_\alpha := f^{-1}((-\infty, \alpha]) \\ G_\alpha := g^{-1}((-\infty, \alpha]) \end{array} \right.$$

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

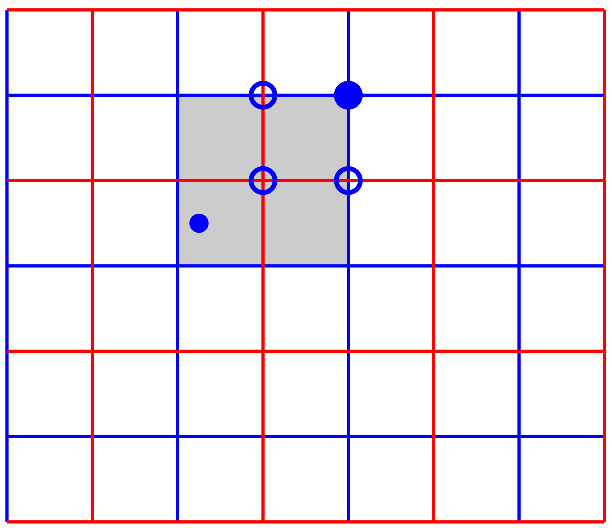
- Back to interleaved filtrations:

$$F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq F_{(2n+2)\varepsilon} \subseteq \cdots$$

$$H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases} \longrightarrow \left(\begin{array}{l} F^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } F \text{ and } H \\ G^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } G \text{ and } H \textbf{ of phase } \varepsilon \end{array} \right.$$

Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\, f, \mathrm{D}_k\, g) \leq 8\varepsilon$

Improvement:



A Simple Proof

$F_\alpha := f^{-1}((-\infty, \alpha])$

$G_\alpha := g^{-1}((-\infty, \alpha])$

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

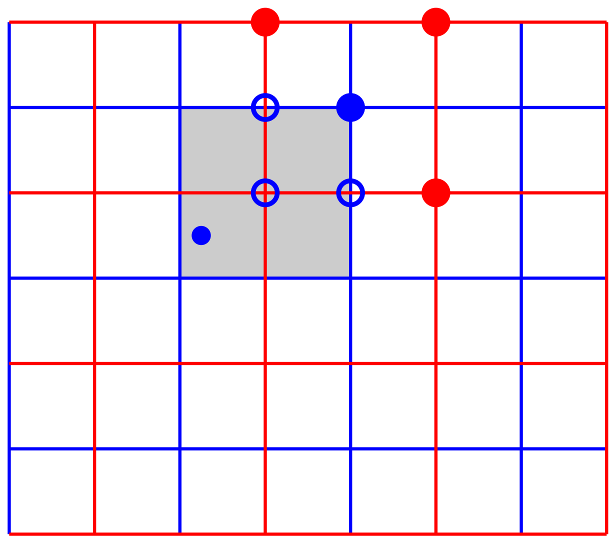
- Back to interleaved filtrations:

$$F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq F_{(2n+2)\varepsilon} \subseteq \cdots$$

$$H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} \text{ if } n \text{ is even} \\ G_{n\varepsilon} \text{ if } n \text{ is odd} \end{cases} \rightarrow \left(\begin{array}{l} F^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } F \text{ and } H \\ G^{2\varepsilon} \text{ is the } 2\varepsilon\text{-pixelization of } G \text{ and } H \textbf{ of phase } \varepsilon \end{array} \right.$$

Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\ f, \mathrm{D}_k\ g) \leq 8\varepsilon$

Improvement:



A Simple Proof

$F_\alpha := f^{-1}((-\infty, \alpha])$

$G_\alpha := g^{-1}((-\infty, \alpha])$

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

- Back to interleaved filtrations:

$$F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq F_{(2n+2)\varepsilon} \subseteq \cdots$$

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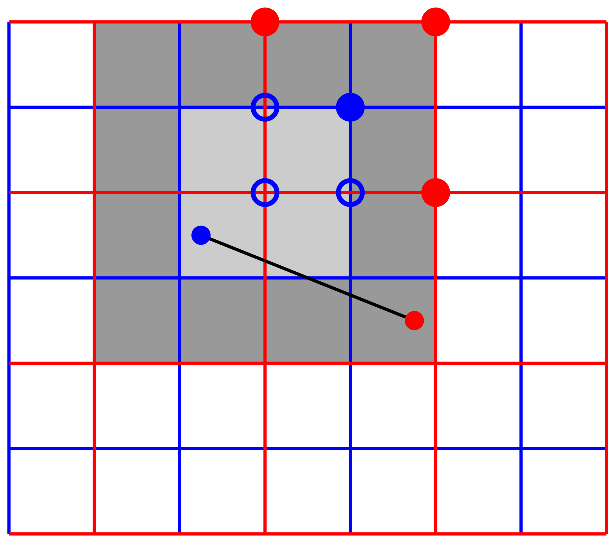
Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\ f, \mathrm{D}_k\ g) \leq 8\varepsilon$

Improvement: $\boxed{\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{D}_k\ f, \mathrm{D}_k\ g) \leq 3\varepsilon}$



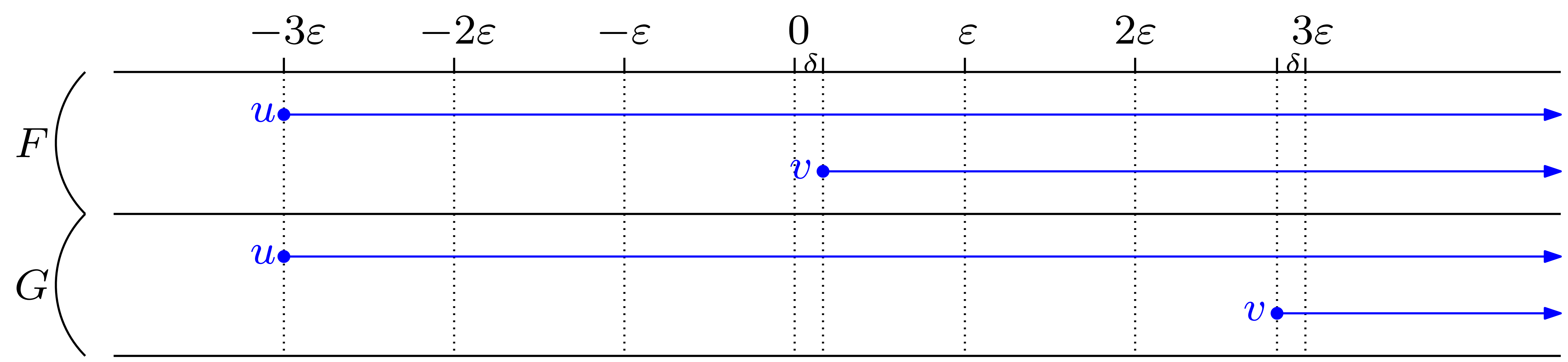
A Simple Proof

$F_\alpha := f^{-1}((-\infty, \alpha])$

$G_\alpha := g^{-1}((-\infty, \alpha])$

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

- Comments:
  - only the fact that F, G are interleaved on a periodic scale $x + \varepsilon\mathbb{Z}$ has been used.
  - under this assumption, the bound is tight:



Outline

- Overview of the proof of [CEH'05] — where continuity and triangulability are needed;

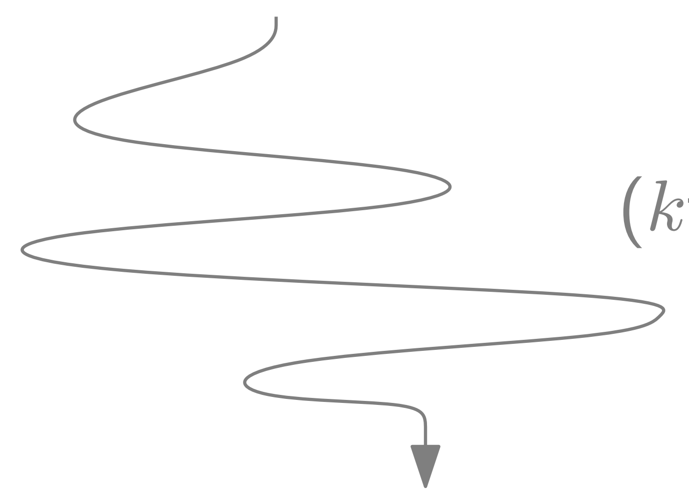
- A new, simple, geometrically-flavored proof of stability with an upper bound of $3\|f-g\|_\infty$ on the bottleneck distance;

- Reducing the upper bound from $3\|f-g\|_\infty$ to $\|f-g\|_\infty$ — interpolating at algebraic level directly.

Working at Algebraic Level

$\longrightarrow \left(\begin{array}{l} \text{give up functional point of view / work at homology level} \\ \text{rephrase statements and proofs in purely algebraic terms} \end{array} \right.$

$$F_\alpha \hookrightarrow G_{\alpha+\varepsilon} \hookrightarrow F_{\alpha+2\varepsilon} \hookrightarrow G_{\alpha+3\varepsilon} \hookrightarrow \cdots$$



(kth homology level)

$$H_k(F_\alpha) \rightarrow H_k(G_{\alpha+\varepsilon}) \rightarrow H_k(F_{\alpha+2\varepsilon}) \rightarrow H_k(G_{\alpha+3\varepsilon}) \rightarrow \cdots$$

Working at Algebraic Level

(Note: all vector spaces are over a same fixed field, omitted in our notations)

A **persistence module** indexed by $\mathbb{R}$ is a family $\{F_\alpha\}_{\alpha\in\mathbb{R}}$ of vector spaces and a family of linear maps $\{f_\alpha^\beta : F_\alpha \to F_\beta\}_{\alpha\leq\beta}$ such that $\forall \alpha \leq \beta \leq \gamma$, $f_\alpha^\alpha = \mathrm{id}_{F_\alpha}$ and $f_\beta^\gamma \circ f_\alpha^\beta = f_\alpha^\gamma$.

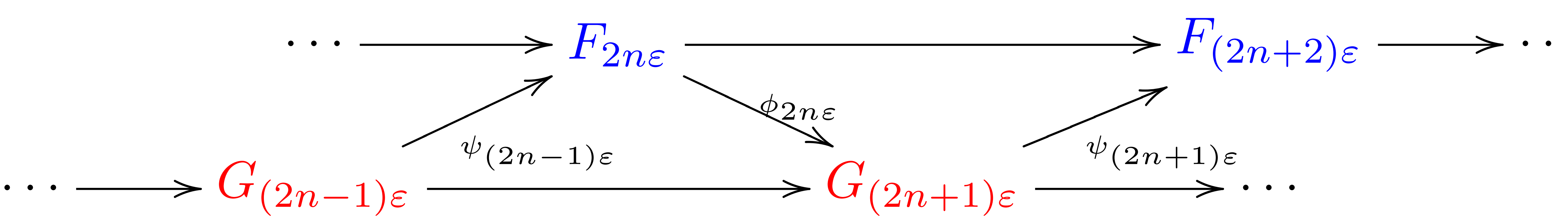
Weak tameness condition: a persistence module $\mathcal{F}_\mathbb{R} = (F_\alpha, f_\alpha^\beta)$ is 0-tame if $\mathrm{rank}\ f_\alpha^\beta < +\infty$ for all $\alpha < \beta$.

Working at Algebraic Level

(Note: all vector spaces are over a same fixed field, omitted in our notations)

A **persistence module** indexed by $\mathbb{R}$ is a family $\{F_\alpha\}_{\alpha\in\mathbb{R}}$ of vector spaces and a family of linear maps $\{f_\alpha^\beta : F_\alpha \to F_\beta\}_{\alpha\leq\beta}$ such that $\forall \alpha \leq \beta \leq \gamma$, $f_\alpha^\alpha = \mathrm{id}_{F_\alpha}$ and $f_\beta^\gamma \circ f_\alpha^\beta = f_\alpha^\gamma$.

Two persistence modules $\mathcal{F}_\mathbb{R} = (F_\alpha, f_\alpha^\beta)$ and $\mathcal{G}_\mathbb{R} = (G_\alpha, g_\alpha^\beta)$ are ε-**interleaved** if there exist two families of homomorphisms: $\{\phi_{n\varepsilon} : F_{n\varepsilon} \to G_{(n+1)\varepsilon}\}_{n\in 2\mathbb{Z}}$ and $\{\psi_{n\varepsilon} : G_{n\varepsilon} \to F_{(n+1)\varepsilon}\}_{n\in 1+2\mathbb{Z}}$ that make the following diagram commute:



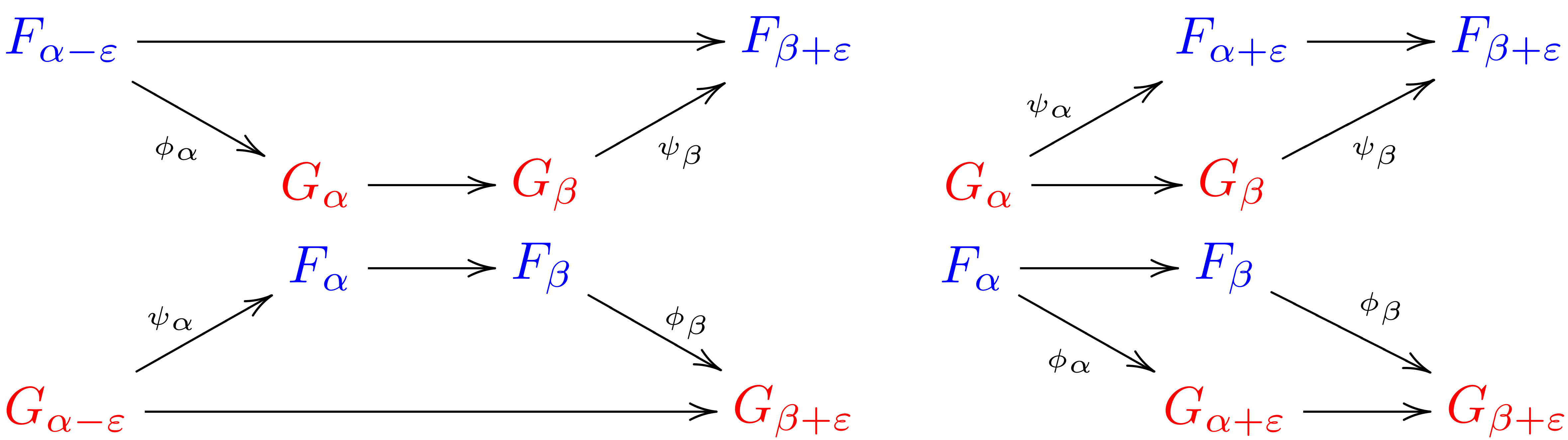
- proof of 3ε bound still holds (mixed module $\{H_{n\varepsilon}\}_{n\in\mathbb{Z}}$ defined similarly)

Working at Algebraic Level

(Note: all vector spaces are over a same fixed field, omitted in our notations)

A **persistence module** indexed by $\mathbb{R}$ is a family $\{F_\alpha\}_{\alpha\in\mathbb{R}}$ of vector spaces and a family of linear maps $\{f_\alpha^\beta : F_\alpha \to F_\beta\}_{\alpha\leq\beta}$ such that $\forall \alpha \leq \beta \leq \gamma$, $f_\alpha^\alpha = \mathrm{id}_{F_\alpha}$ and $f_\beta^\gamma \circ f_\alpha^\beta = f_\alpha^\gamma$.

Two persistence modules $\mathcal{F}_\mathbb{R} = (F_\alpha, f_\alpha^\beta)$ and $\mathcal{G}_\mathbb{R} = (G_\alpha, g_\alpha^\beta)$ are **strongly** ε-**interleaved** if $\exists$ two families of homomorphisms: $\{\phi_\alpha : F_\alpha \to G_{\alpha+\varepsilon}\}_{\alpha\in\mathbb{R}}$ and $\{\psi_\alpha : G_\alpha \to F_{\alpha+\varepsilon}\}_{\alpha\in\mathbb{R}}$ that make the following diagrams commute $\forall \alpha \leq \beta$:

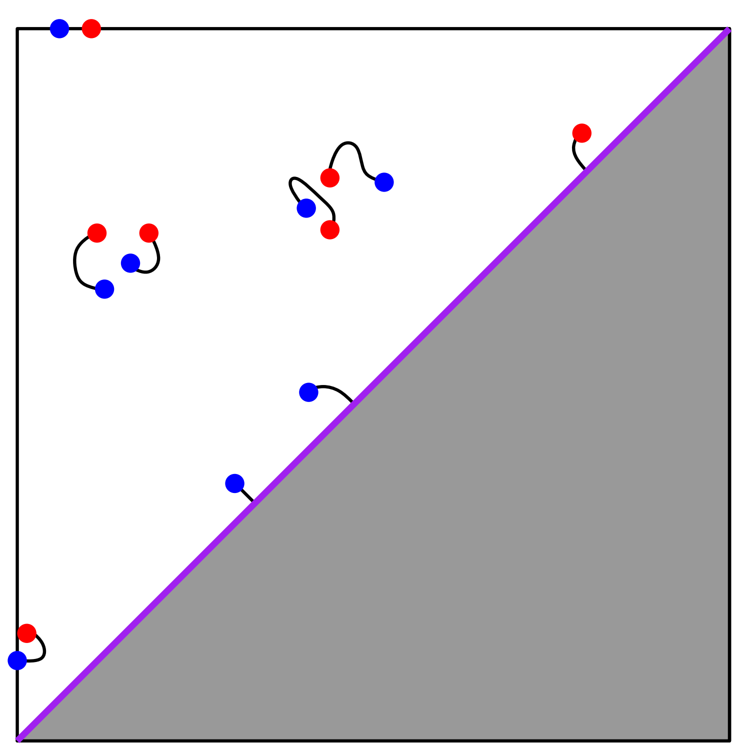


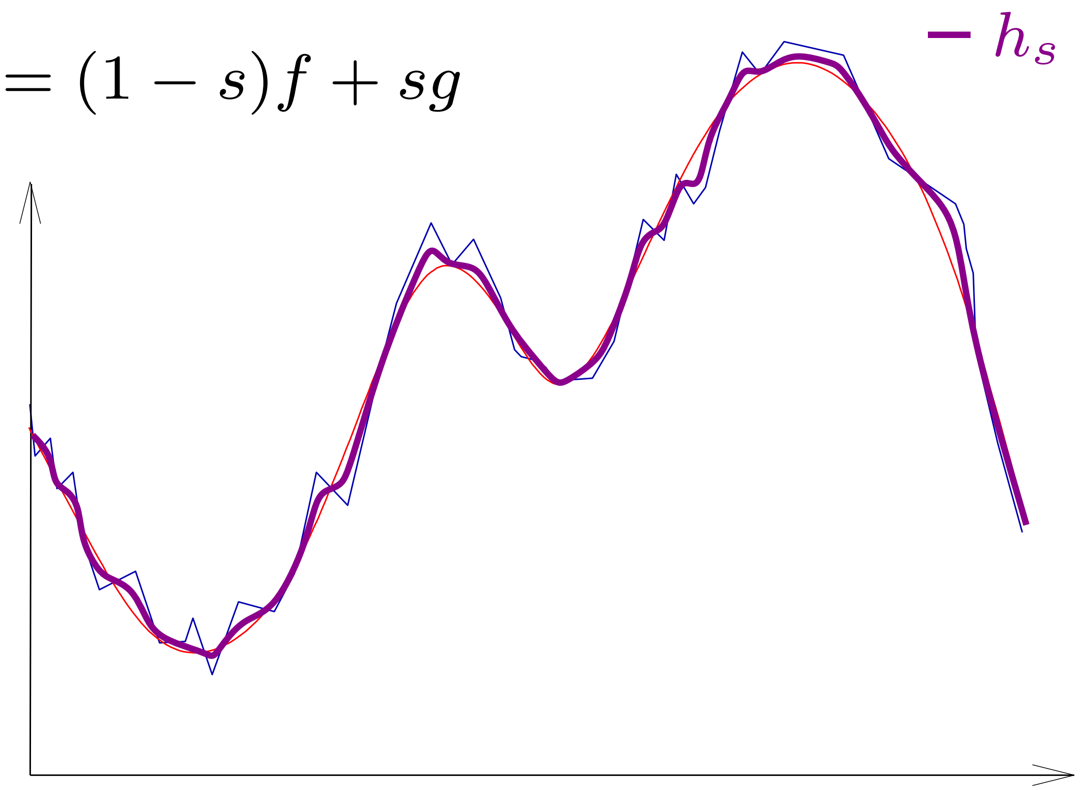
Back to the Original Proof

Let $f, g : \mathbb{X} \to \mathbb{R}$ be tame, and let $\varepsilon = \|f - g\|_\infty$.

$$F_\alpha := f^{-1}((-\infty, \alpha])$$
$$G_\alpha := g^{-1}((-\infty, \alpha])$$

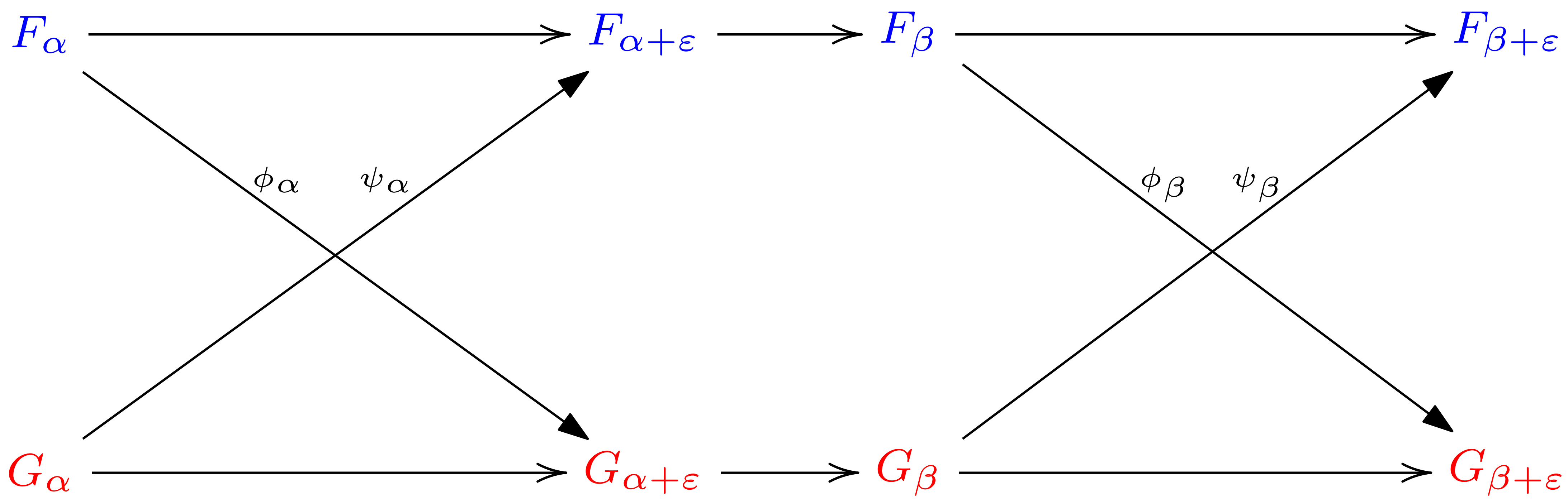
- Key observation: $\{F_\alpha\}_\alpha$ and $\{G_\alpha\}_\alpha$ are ε-**interleaved** w.r.t. inclusion:
- Bound on Hausdorff distance: $\mathrm{d}_{\mathrm{H}}^{\infty}(\mathrm{D}_k\, f, \mathrm{D}_k\, g) \leq \varepsilon$
- From Hausdorff to bottleneck: the *infinitesimal* case:
- Interpolation argument: $\forall s \in [0,1]$, $h_s = (1-s)f + sg$





Interpolation between Persistence Modules

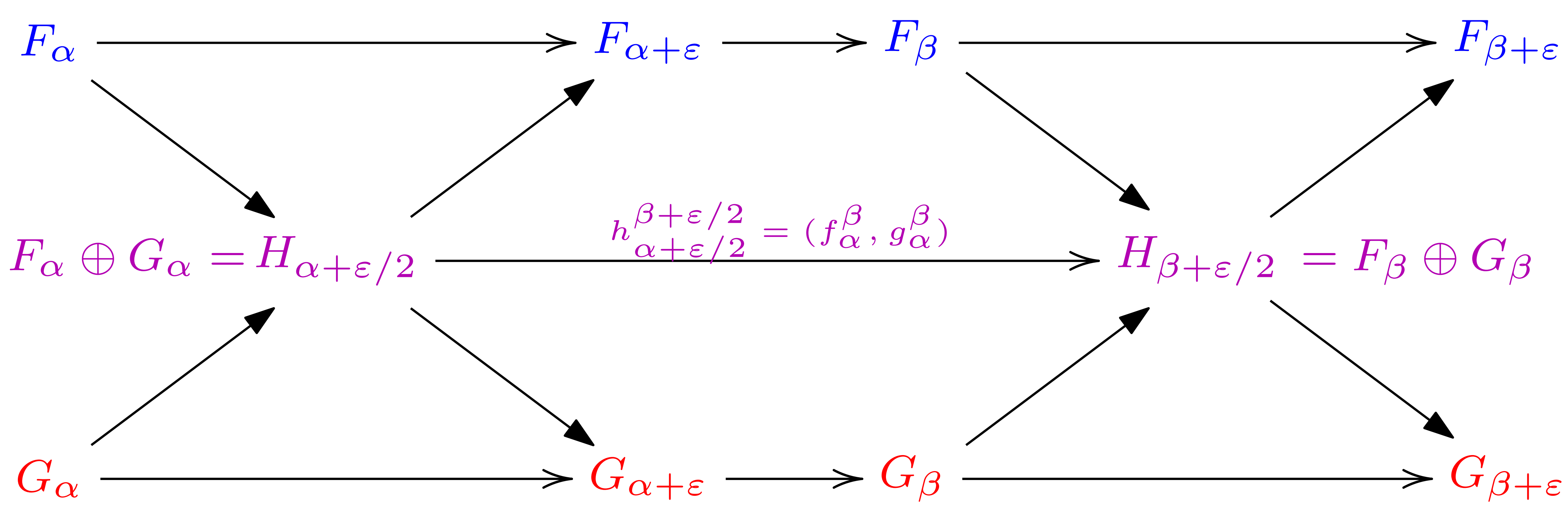
Let $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ be strongly ε-interleaved.



Interpolation between Persistence Modules

Let $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ be strongly ε-interleaved.

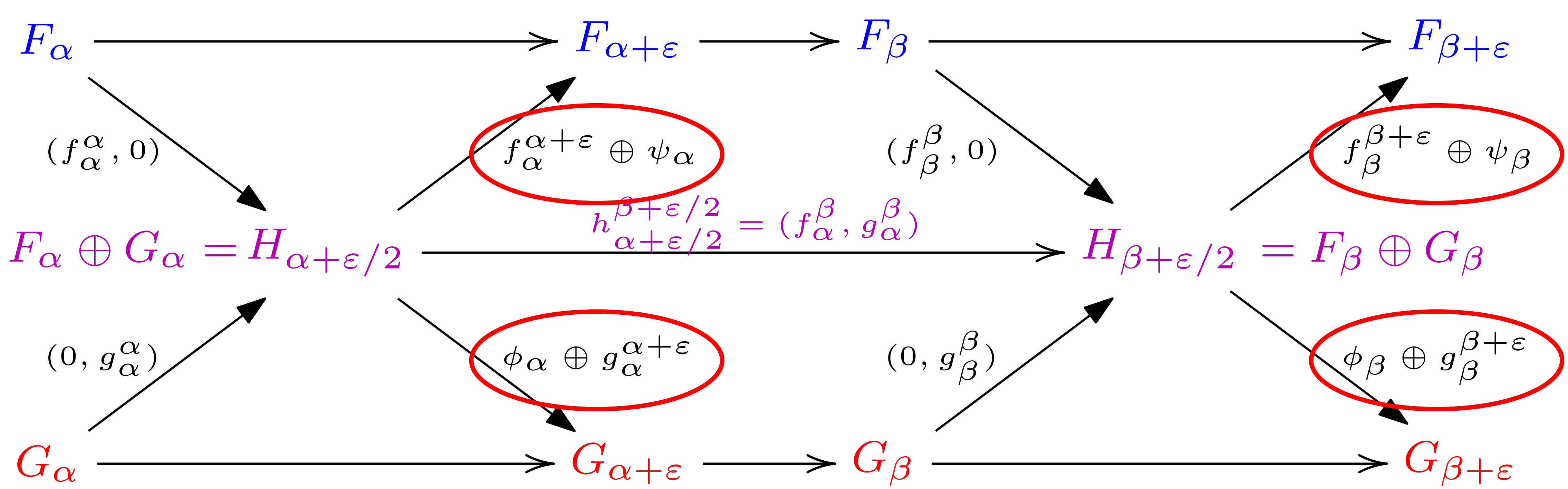
- define $\mathcal{H}_{\mathbb{R}}$ as the $\frac{\varepsilon}{2}$-shifted direct sum of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$,



Interpolation between Persistence Modules

Let $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ be strongly ε-interleaved.

- define $\mathcal{H}_{\mathbb{R}}$ as the $\frac{\varepsilon}{2}$-shifted direct sum of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$,
- combine both coordinates when projecting back onto $\mathcal{F}_{\mathbb{R}}$ or $\mathcal{G}_{\mathbb{R}}$,



Interpolation between Persistence Modules

Let $\mathcal{F}_\mathbb{R}$ and $\mathcal{G}_\mathbb{R}$ be strongly ε-interleaved.

- define $\mathcal{H}_\mathbb{R}$ as the $\frac{\varepsilon}{2}$-shifted direct sum of $\mathcal{F}_\mathbb{R}$ and $\mathcal{G}_\mathbb{R}$,
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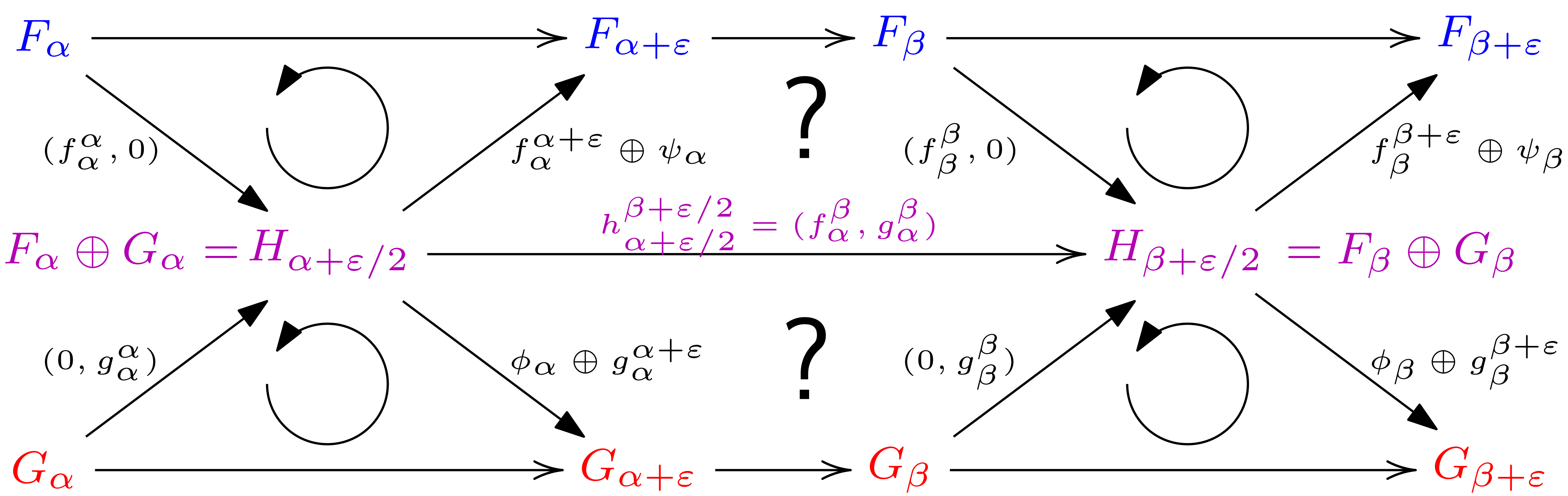
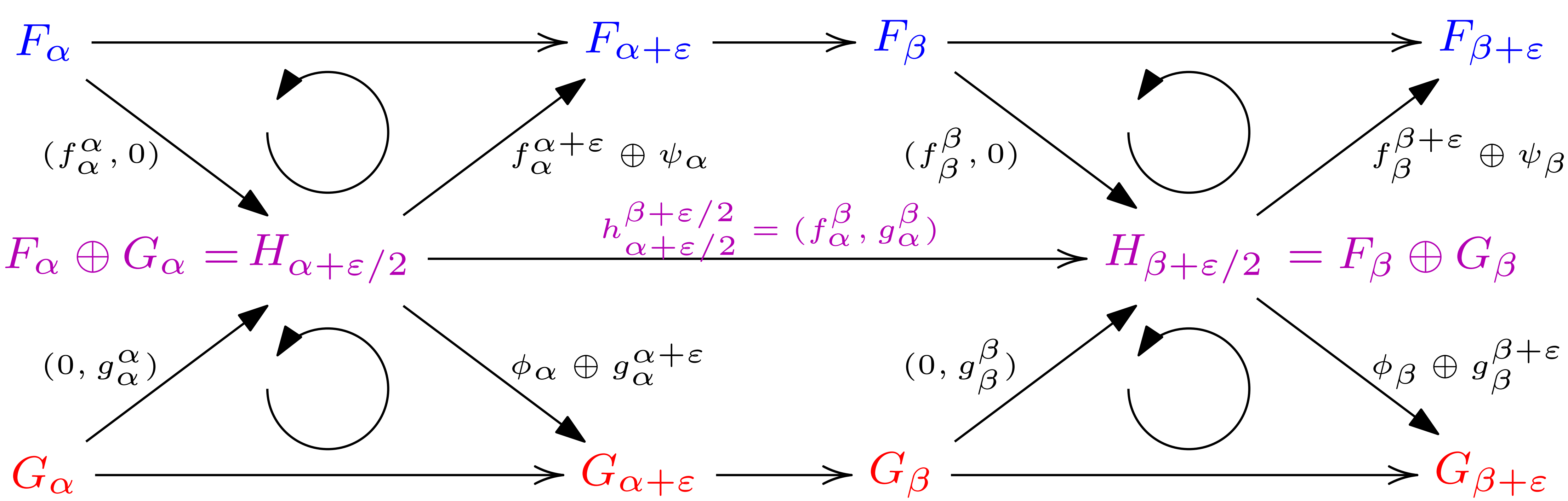


Interpolation between Persistence Modules

Let $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ be strongly ε-interleaved.

- define $\mathcal{H}_{\mathbb{R}}$ as the $\frac{\varepsilon}{2}$-shifted direct sum of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$,
- combine both coordinates when projecting back onto $\mathcal{F}_{\mathbb{R}}$ or $\mathcal{G}_{\mathbb{R}}$,
- make trapezoids commute: $\forall \beta \geq \alpha + \varepsilon$, $\forall (x,y) \in F_\alpha \oplus G_\alpha$,

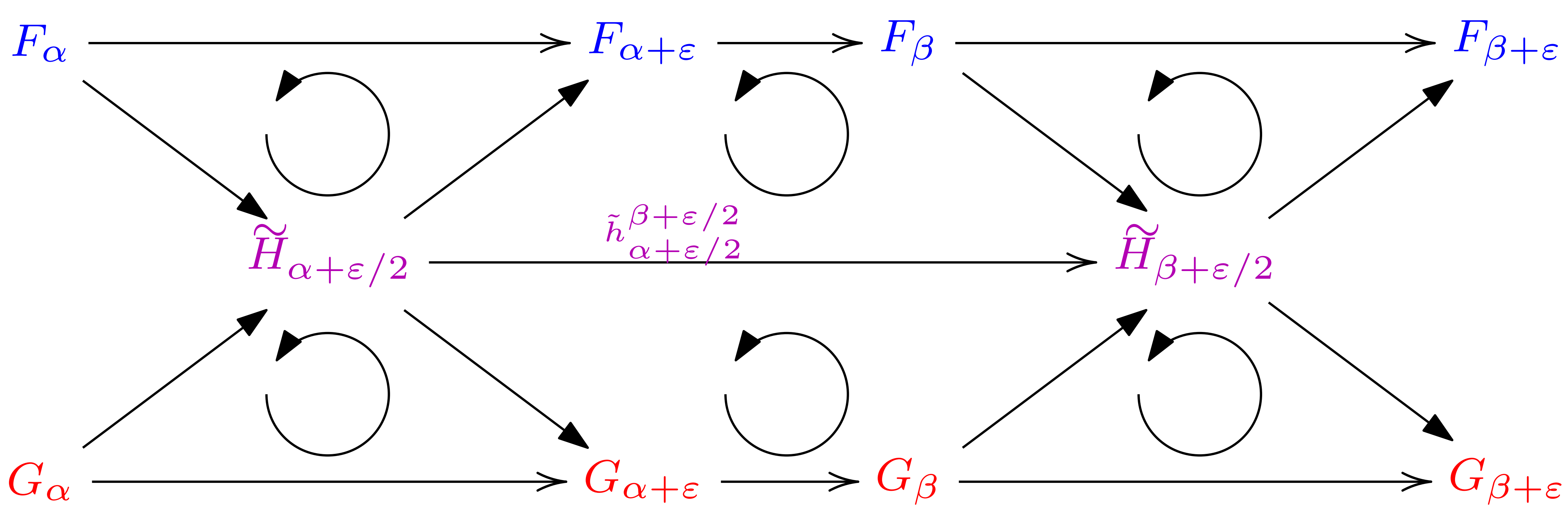
identify $(f_\alpha^\beta(x) + f_{\alpha+\varepsilon}^\beta \circ \psi_\alpha(y), 0)$ with $(f_\alpha^\beta(x), g_\alpha^\beta(y))$



Interpolation between Persistence Modules

Let $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ be strongly ε-interleaved.

Quotient persistence module $(\tilde{H}_\alpha, \tilde{h}_\alpha^\beta)$ is *midpoint* of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$.

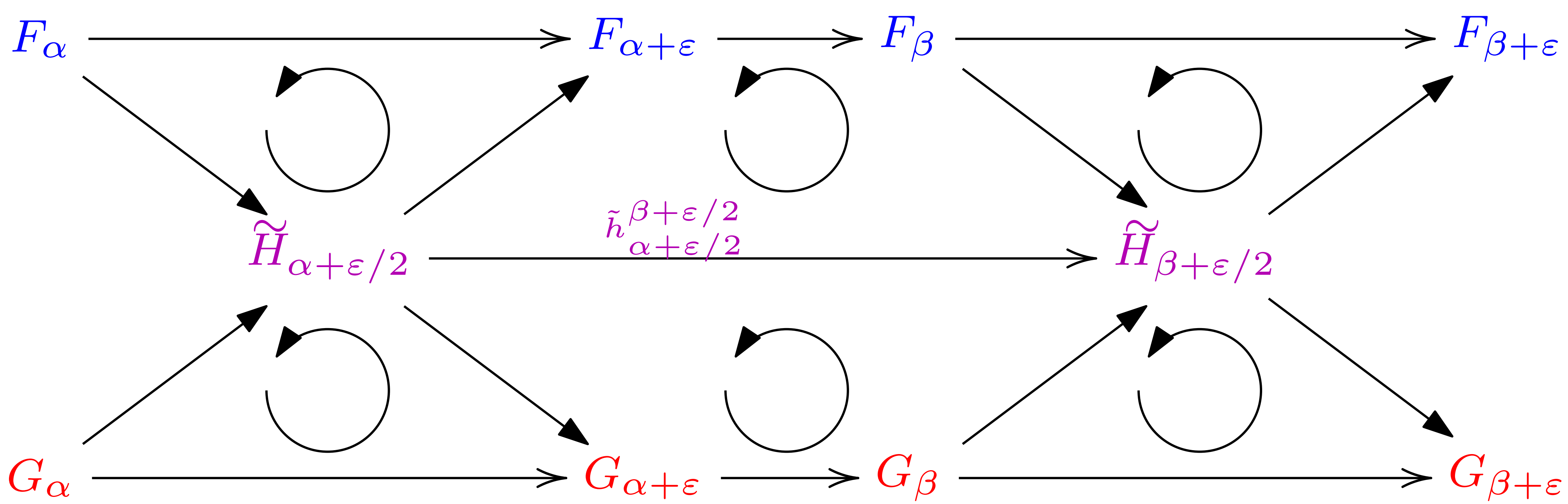


Interpolation between Persistence Modules

Let $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ be strongly ε-interleaved.

Quotient persistence module $(\tilde{H}_\alpha, \tilde{h}_\alpha^\beta)$ is *midpoint* of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$.

More generally, we can define any *convex combination* of $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$.



Interpolation between Persistence Modules

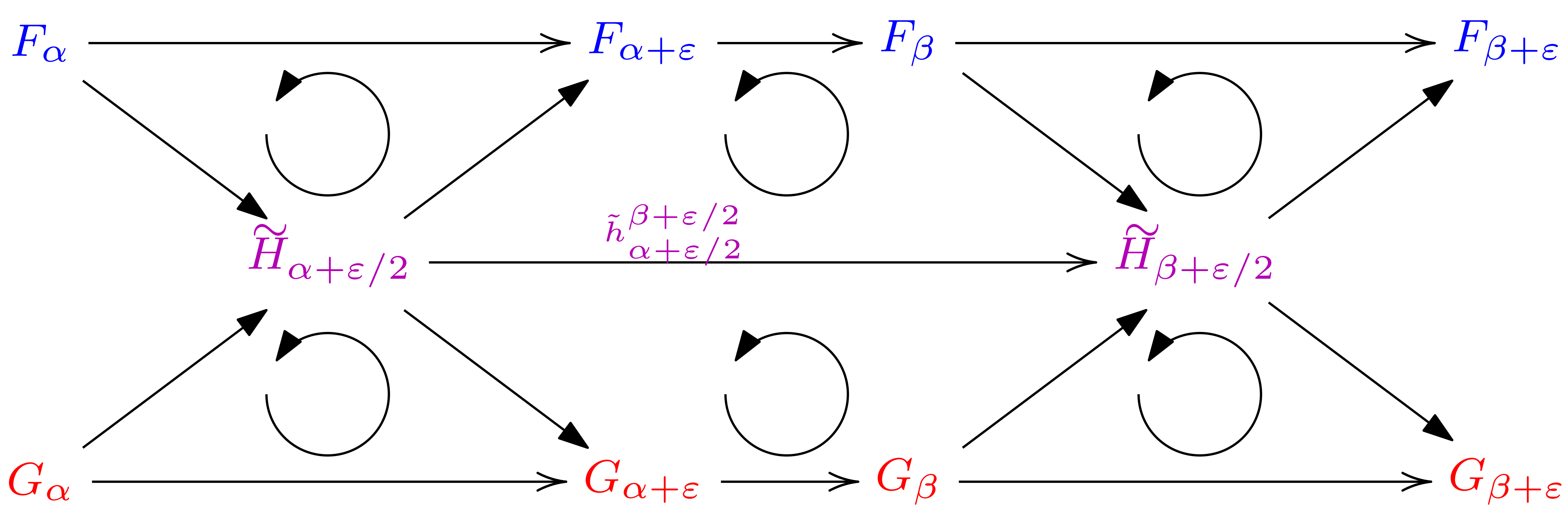
Let $\mathcal{F}_\mathbb{R}$ and $\mathcal{G}_\mathbb{R}$ be strongly ε-interleaved.

Quotient persistence module $(\tilde{H}_\alpha, \tilde{h}_\alpha^\beta)$ is *midpoint* of $\mathcal{F}_\mathbb{R}$ and $\mathcal{G}_\mathbb{R}$.

More generally, we can define any *convex combination* of $\mathcal{F}_\mathbb{R}$ and $\mathcal{G}_\mathbb{R}$.

Interpolation argument of [CEH'05] applies straightforwardly.

$\Rightarrow \mathrm{d}_\mathrm{B}^\infty(\mathrm{D}\,\mathcal{F}_\mathbb{R}, \mathrm{D}\,\mathcal{G}_\mathbb{R}) \leq \varepsilon$



Take-Home Message(s)

- Stability is central in topological data analysis and simplification.

- We provide stability results for larger classes of spaces and functions.

- Basic version of our proof is simple and geometrically-flavored.

- Rephrasing of our results in a purely algebraic context enables the comparison of functions defined over different spaces (cf. Primoz's talk).