2D order-k line-space Voronoi diagram

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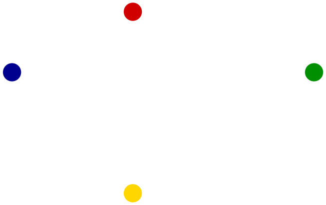
JGA 2009

Introduction

Given *n* sites (points) in the plane,

- the Voronoi diagram is the partition of the points of the plane
- according to the site(s) they are nearest to

- the 2D line space Voronoi Diagram is the partition of the lines of the plane
- according to the site(s) they are nearest to



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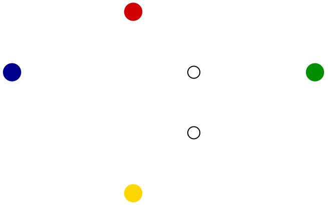

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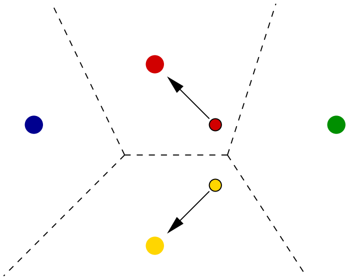

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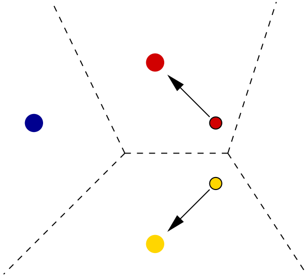

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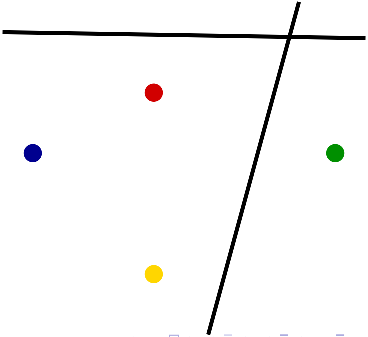

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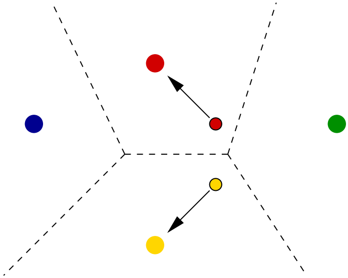

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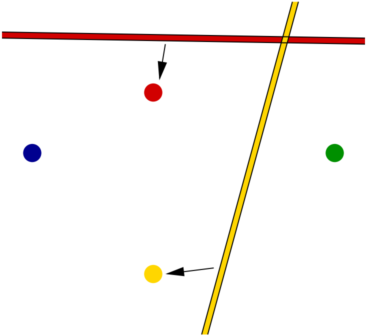

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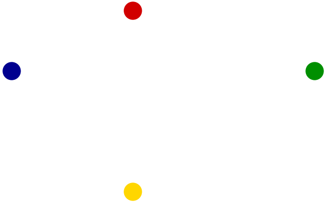

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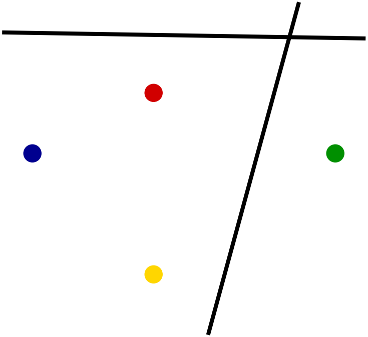

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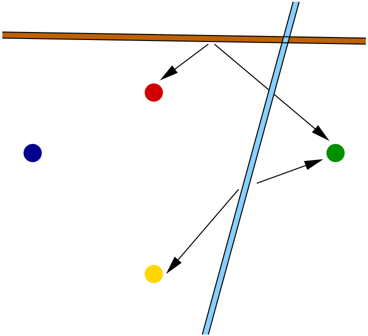


Outline

The 2D order-k line space Voronoi Diagram

Definition

Given n sites (points) in the plane, the 2D order-k line space Voronoi Diagram is the partition of the set of lines of the plane according to the k sites they are nearest to.

It is a partition into:

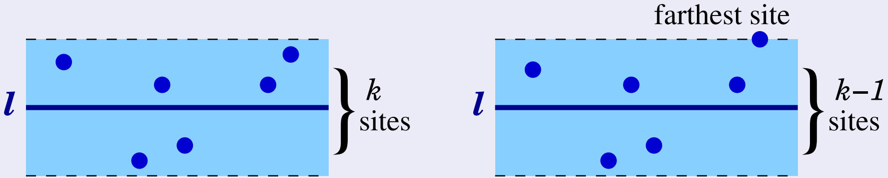
- faces: lines closer to k sites
- edges: lines equidistant to two sites and closer to $k-1$ another sites
- vertices: lines equidistant to three sites and closer to $k-1$ (resp. $k-2$) another sites

Elements of the diagram

Faces

A line l is closer to k site $(p_{i_1}, \ldots, p_{i_k})$

- $\iff$ there exists a strip centered in l containing only the k sites
- $\iff$ the strip centered in l passing through the farthest site contains only the $k-1$ other ones



Such a line belongs to a face $f^k(p_{i_1}, \ldots, p_{i_k})$ of the diagram

Elements of the diagram

Edges

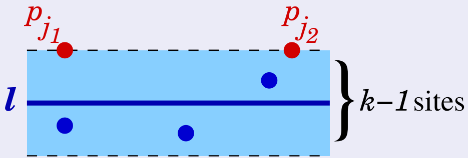
- A line passes
  – from the face $f^k(p_{i_1}, \ldots, p_{i_{k-1}}, p_{j_1})$
  – to the face $f^k(p_{i_1}, \ldots, p_{i_{k-1}}, p_{j_2})$
  when the farthest point p_{j_1} is replaced by another one p_{j_2} (different from the other $k-1$ sites)
- by crossing the edge $e^k((p_{i_1}, \ldots, p_{i_{k-1}}), (p_{j_1}, p_{j_2}))$, when p_{j_1} and p_{j_2} are the farthest equidistant sites
- A line l belongs to an edge $\iff$ the strip centered in l passing through p_{j_1} and p_{j_2} contains only the other $k-1$ sites

Elements of the diagram

Edges

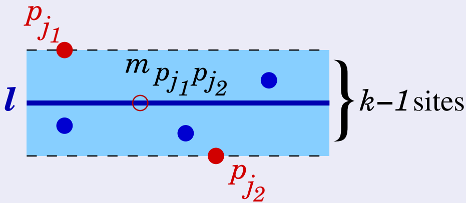
- p_{j_1} and p_{j_2} are on the same border of the strip
  $\Rightarrow l$ is parallel to (p_{j_1}, p_{j_2})

$$e^k_{//}((p_{i_1}, \dots, p_{i_{k-1}}), (p_{j_1}, p_{j_2}))$$



- p_{j_1} and p_{j_2} are on the opposite sides of the strip
  $\Rightarrow l$ passes through the midpoint $m_{p_{j_1} p_{j_2}}$

$$e^k_{m}((p_{i_1}, \dots, p_{i_{k-1}}), (p_{j_1}, p_{j_2}))$$



Elements of the diagram

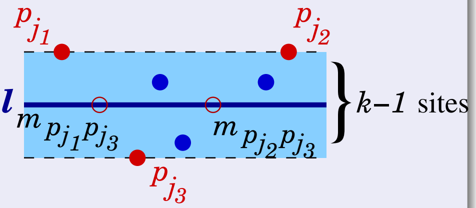
Vertices

- A line passes
  – from the edge $e^k((p_{i_1}, \ldots, p_{i_{k-1}}), (p_{j_1}, p_{j_2}))$
  – to the edge $e^k((p_{i_1}, \ldots, p_{i_{k-1}}), (p_{j_1}, p_{j_3}))$
  when the farthest point p_{j_2} is replaced by another one p_{j_3} :
  (1) different from the other $k-1$ sites
  (2) belonging to the other $k-1$ sites
- by crossing the vertex
  (1) $v^k_{-1}((p_{i_1}, \ldots, p_{i_{k-1}}), (p_{j_1}, p_{j_2}, p_{j_3}))$
  (2) $v^k_{-2}((p_{i_1}, \ldots, p_{i_{k-2}}), (p_{j_1}, p_{j_2}, p_{i_{k-1}}))$
  when p_{j_1}, p_{j_2}, and p_{j_3} are the farthest equidistant sites
- A line l is a vertex $\iff$ the strip centered in l passing through p_{j_1}, p_{j_2}, and p_{j_3}
  (1) contains only the other $k-1$ sites
  (2) contains only the other $k-2$ sites

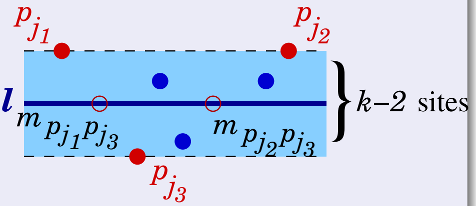
Elements of the diagram

Vertices

$v^k_{-1}((p_{i_1}, \ldots, p_{i_{k-1}}), (p_{j_1}, p_{j_2}, p_{j_3}))$



$v^k_{-2}((p_{i_1}, \ldots, p_{i_{k-2}}), (p_{j_1}, p_{j_2}, p_{j_3}))$



- l is parallel to (p_{j_1}, p_{j_2}) and passes through the midpoints $m_{p_{j_1} p_{j_3}}$ and $m_{p_{j_2} p_{j_3}}$
  $\Rightarrow$ a vertex is incident to one $e^k_{//}$ edge and to two e^k_m edges

Elements of the diagram

Vertices

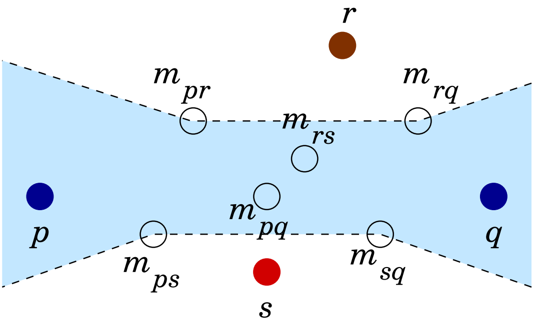
$v^1_{-2}(p_{j_1}, p_{j_2})$

- l is parallel to (p_{j_1}, p_{j_2}) and passes through the midpoint $m_{p_{j_1} p_{j_2}}$
  $\Rightarrow$ a v^1_{-2} vertex is incident to two $e^k_{//}$ edges and to two e^k_m edges

The 2D line space Voronoi Diagram in the scene

Exemples of elements of the 2D line space Voronoi Diagram:

- Face
- Edges
- Vertex
- Faces and incidents edges and vertex



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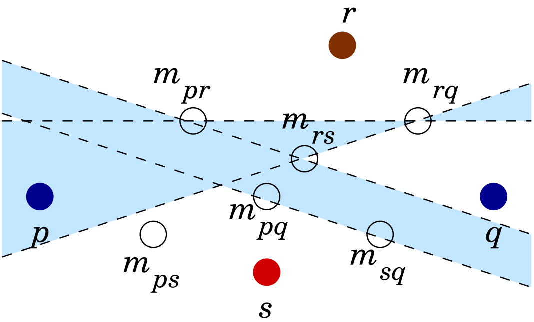

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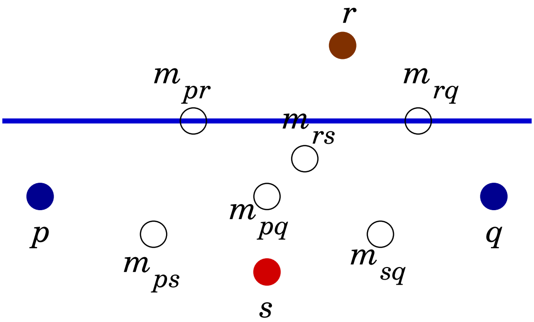

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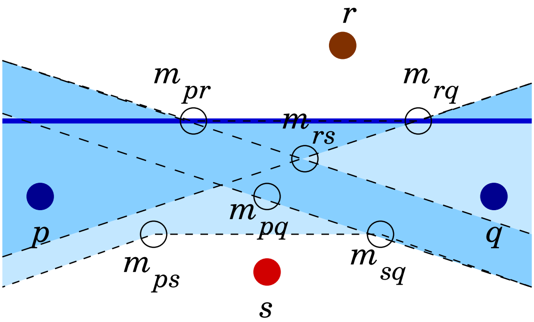
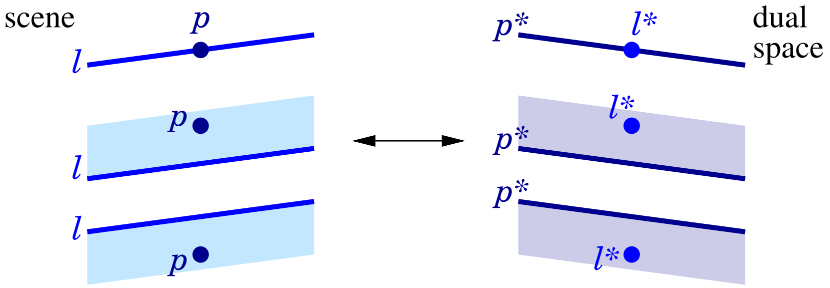


Duality

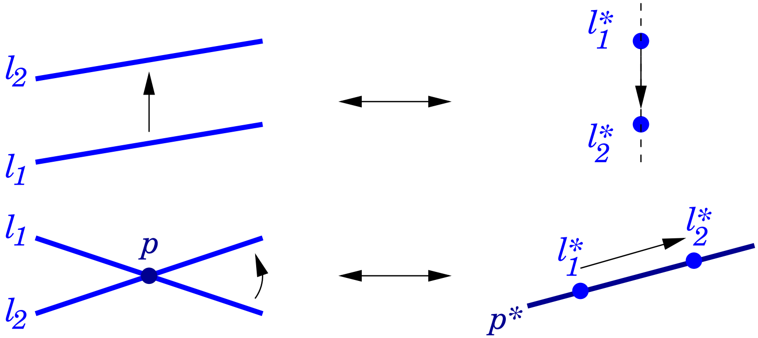
- Sets of lines and their incidence relationships are difficult to see directly
- We use a duality relationship between lines and points: a bijection which transforms a line l into a point l^* and a point p into a line p^* in the dual space such that incidence relationships are preserved (or reversed): $p \in l^{\pm} \iff l^* \in p^{*\mp}$



Duality (cont.)

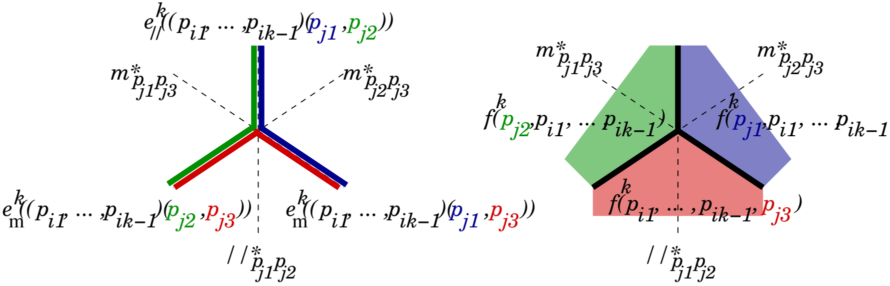
We use the following duality relationship:

- $l : y = ax + b \rightarrow l^*(a, -b)$
- $p(a, b) \rightarrow p^* : y = ax - b$



The line space Voronoi diagram in dual space

Around a v_{-1} vertex

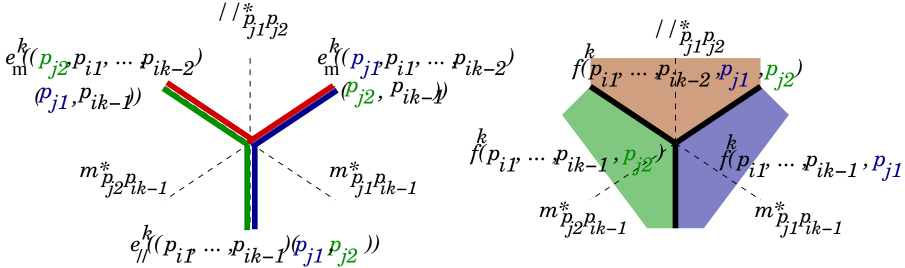


vertex $v^k_{-1}((p_{i1}, \dots, p_{ik-1})(p_{j1}, p_{j2}, p_{j3}))$

A v_{-1} vertex is "Y" shaped, incident to one $e_{//}$ edge and two e_m edges

The line space Voronoi diagram in dual space

Around a v_{-2} vertex

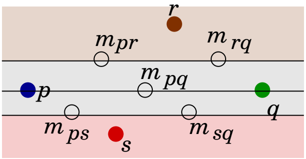


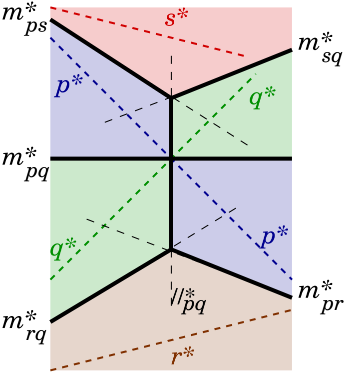
vertex $v^{k}_{-2}((p_{i1}, \ldots, p_{ik-2})(p_{ik-1}, p_{j2}, p_{j3}))$

A v_{-2} vertex is "Y" shaped, incident to one $e_{//}$ edge and two e_m edges

The line space Voronoi diagram in dual space

Around a v^1_{-2} vertex

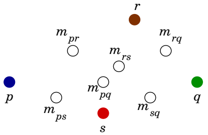


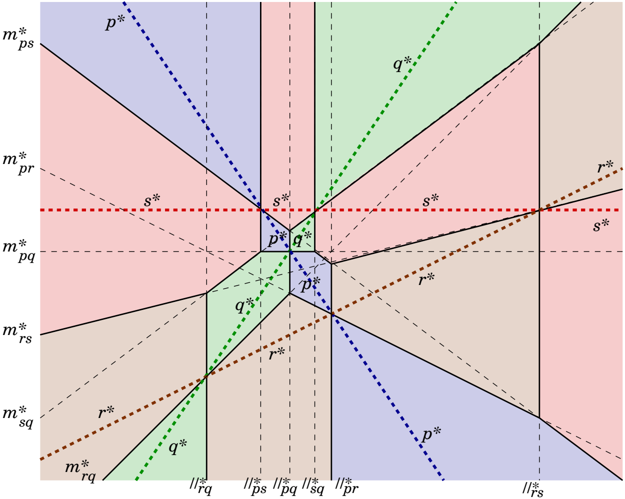


A $v^1_{-2}(pq)$ vertex is the intersection of $//^*_{pq}$ and m^*_{pq}

The line space Voronoi diagram in dual space

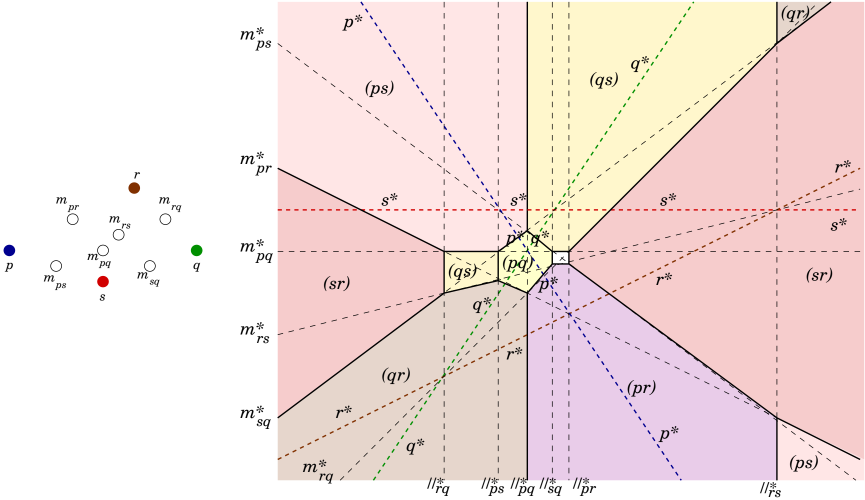
Exemple of a order-1 line space Voronoi diagram





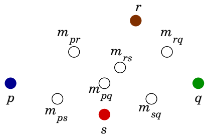
The line space Voronoi diagram in dual space

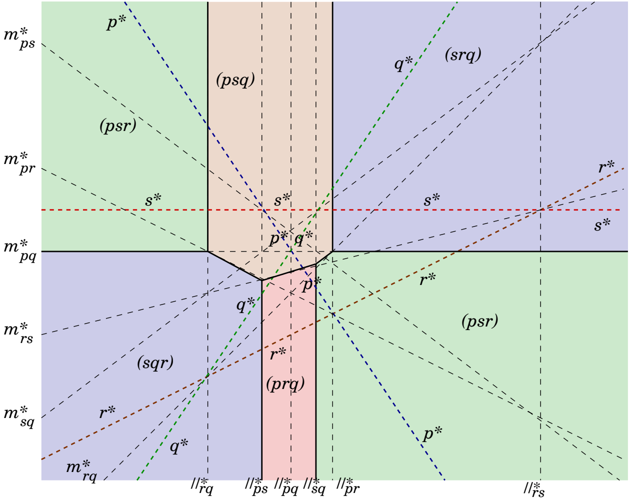
Exemple of a order-2 line space Voronoi diagram



The line space Voronoi diagram in dual space

Exemple of a order-3 line space Voronoi diagram

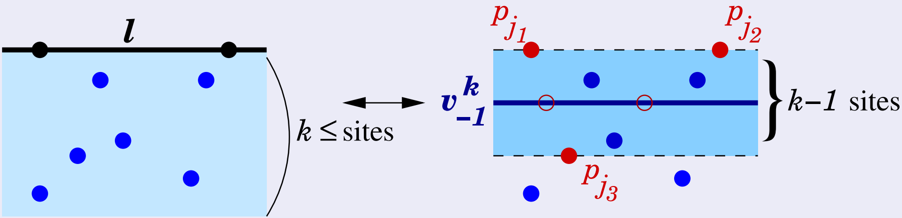




k-sets and order-k line space Voronoi

k-sets and order-k line space Voronoi

- a $k\text{-set}$ if a set of k sites separable from the others by a line
- let L_k be the set of lines passing through two sites having exaclty k sites on one of its side : $|L_k| = |(k+1)\text{-set}|$
- $\sum_1^{n-1} |i\text{-set}| = n(n-1)$ and $\sum_1^{k} |i\text{-set}| = \Theta(nk)$
- a line $l \in L_m$ corresponds to a v^k_{-1} (resp. v^k_{-2}) vertex for $m \geq k$ (resp. $m \geq k-1$) :



The order-k line space Voronoi diagram in dual space

Size

- The order-k diagram of n sites has

$n(n-1) - \sum_1^{k-1} \lvert i\text{-set} \rvert$	v_{-2}^k	vertices
$n(n-1) - \sum_1^{k} \lvert i\text{-set} \rvert$	v_{-1}^k	vertices
$n(n-1) - \sum_1^{k-1} \lvert i\text{-set} \rvert$	$e_{//}^k$	edges
$2(n(n-1) - \sum_1^{k-1} \lvert i\text{-set} \rvert) - \lvert k\text{-set} \rvert + n-k$	e_m^k	edges
$n(n-1) - \sum_1^{k-1} \lvert i\text{-set} \rvert + n-k+1$		faces

- Its size varies from quadratic (order-1) to linear (order-$n-1$)

Computation

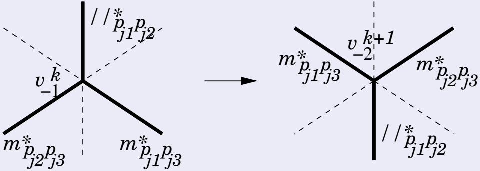
- We can perform a topological sweep of the order-k diagram in $O(n^2)$ time thanks to a topological sweep of the dual line arrangement of the sites.
- which is optimal for $k \leq (n+1)/2$

Order-k and order-$k+1$ line space Voronoi diagram

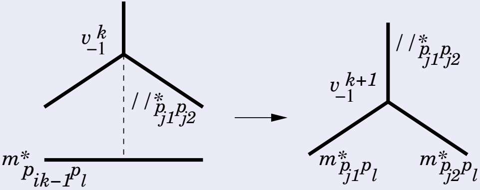
Computing the order-$k+1$ diagram from the order-k one

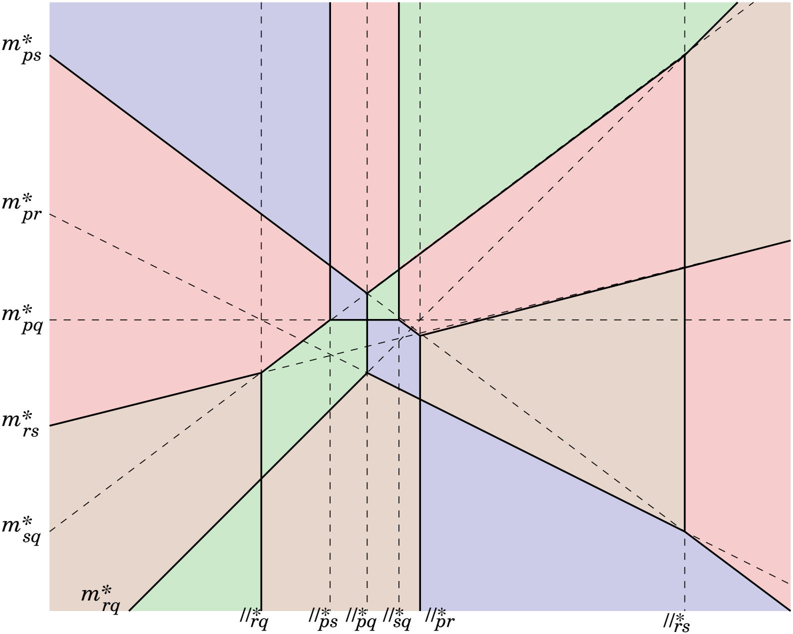
Vertices of the order-$k+1$ diagram can be obtained from the ones of the order-k diagram :

- v_{-2}^{k+1} vertices are obtained by "inverting" v_{-1}^{k} vertices



- new v_{-1}^{k+1} vertices are created from some v_{-1}^{k} vertices

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m^*_{pr}
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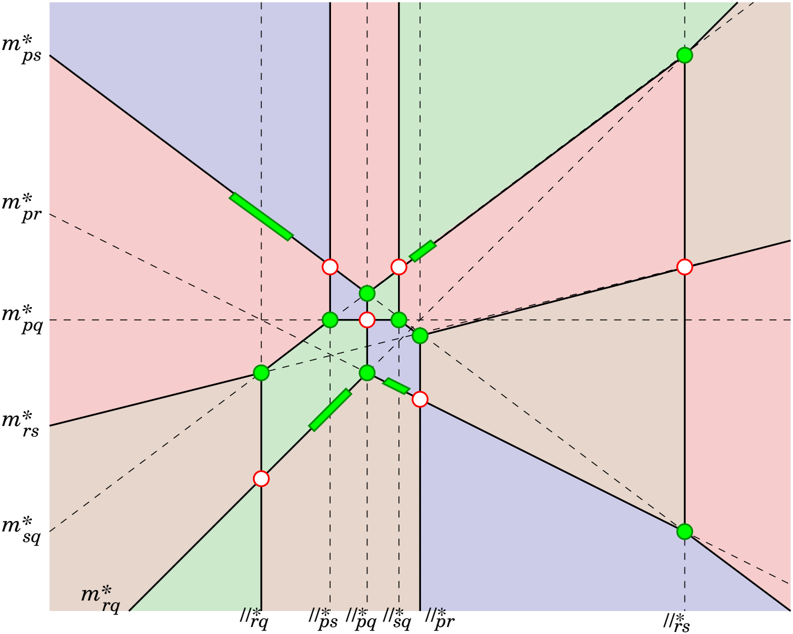
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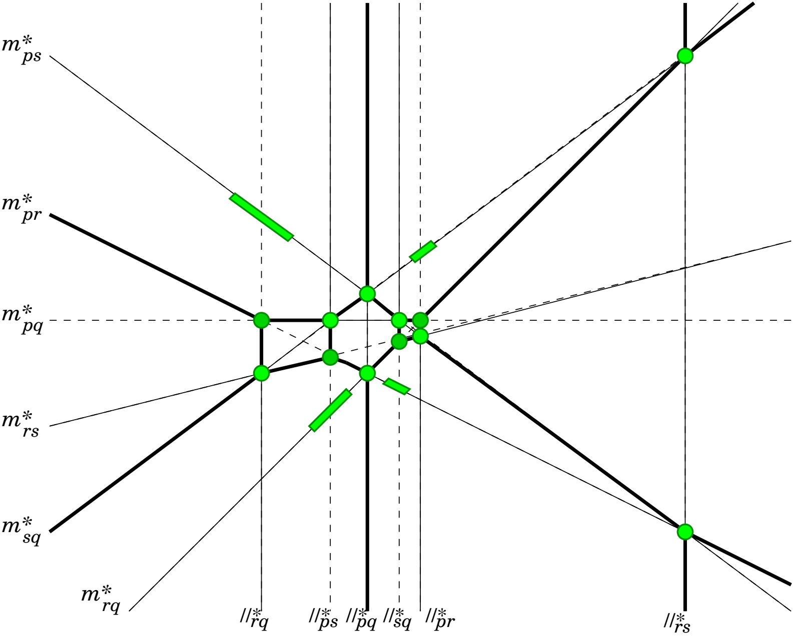
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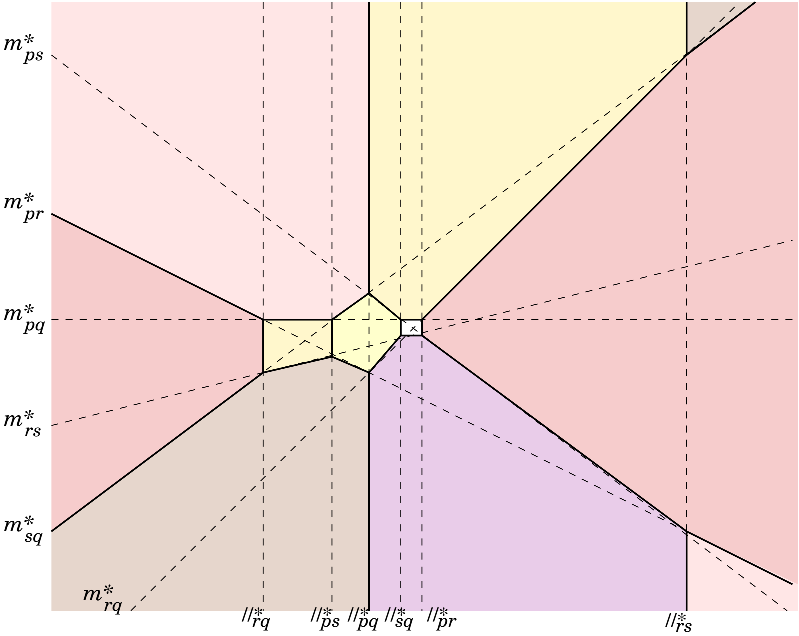

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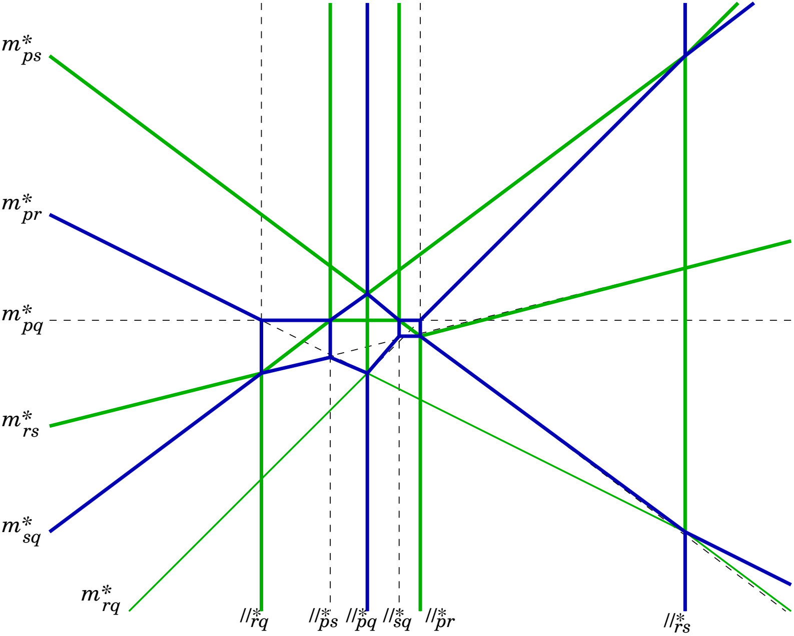

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Applications and perspectives

Some applications

- finding the k nearest sites to a given line in $O(\log n)$ time
- finding the largest k-dense corridor in $O(n^2)$ time and $O(n)$ working space

Perspectives

- dealing with degeneracies, numerical precision
- study of other generalizations of the line space Voronoi Diagram
  - sites other than points,
  - taking visibility into account, ...