Topology in the 21st Century

Vin de Silva
Pomona College

Topology

What is topology?

What is topology?

It is the branch of mathematics which does not distinguish between a teacup and a bagel

one popular answer

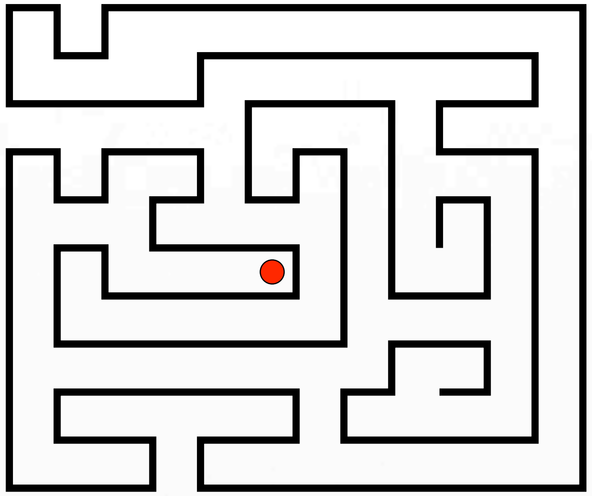
=

Digression: what is topology?

Topology gives answers to qualitative questions

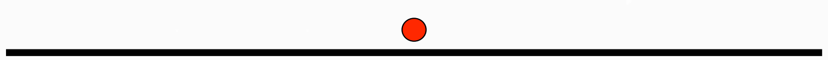
Digression: what is topology?

Topology gives answers to qualitative questions



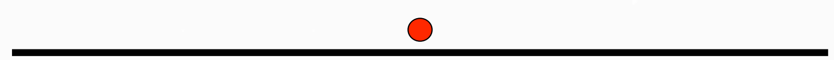
Digression: what is topology?

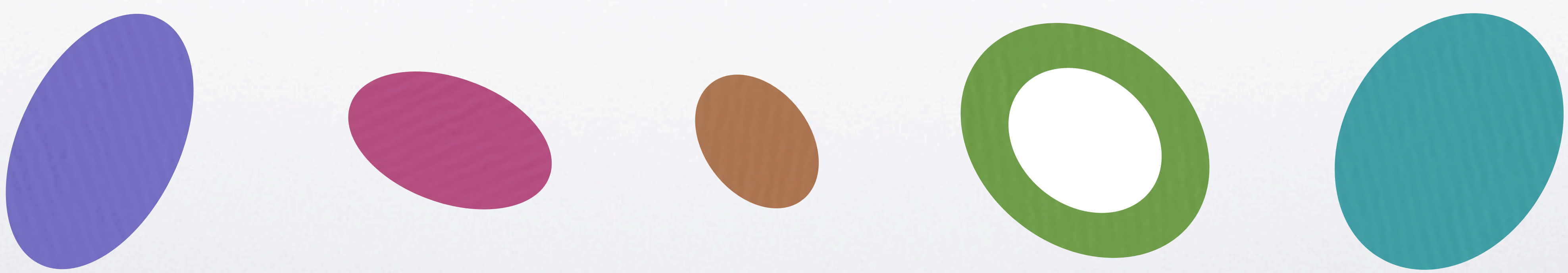
Topology gives answers to qualitative questions



Digression: what is topology?

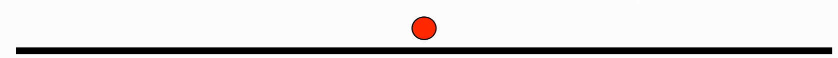
Topology gives answers to qualitative questions

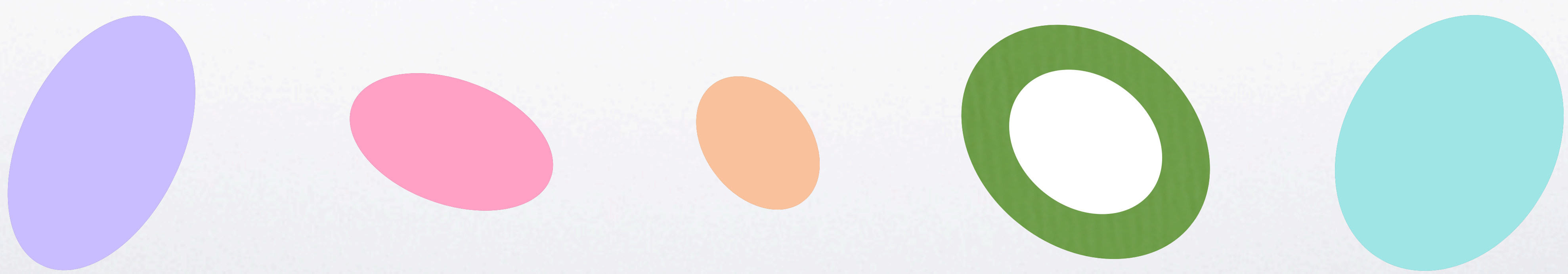




Digression: what is topology?

Topology gives answers to qualitative questions

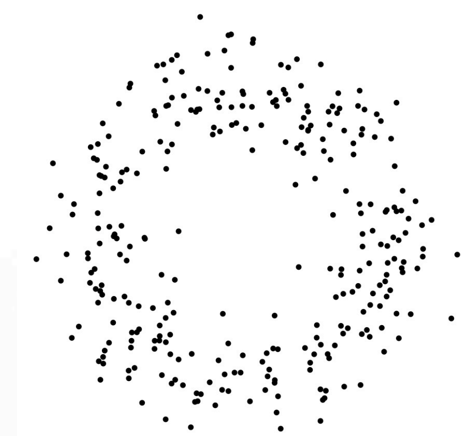


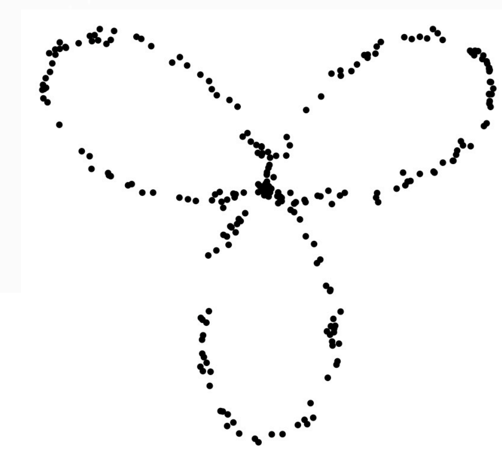


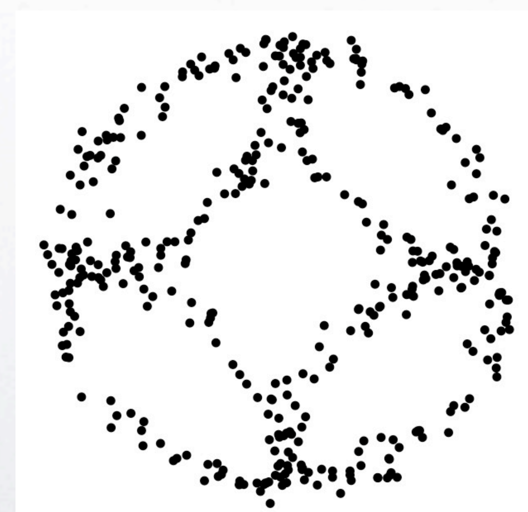
Vin de Silva
http://pages.pomona.edu/~vds04747/

Point-cloud topology

- Topological structure in statistical data
  - density estimation and modality
  - approximation by simplicial complexes
- Assume data have been sampled from some unknown space
  - can we measure topological features of the hidden space?
  - can we assign confidence values to these measurements?
- What does "topology" mean for a cloud of data points?
  - persistent homology
  - spectral theory for point clouds

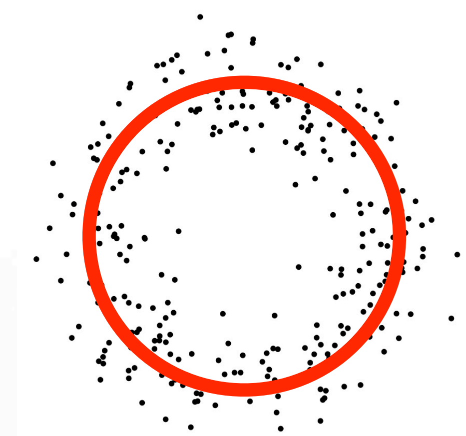


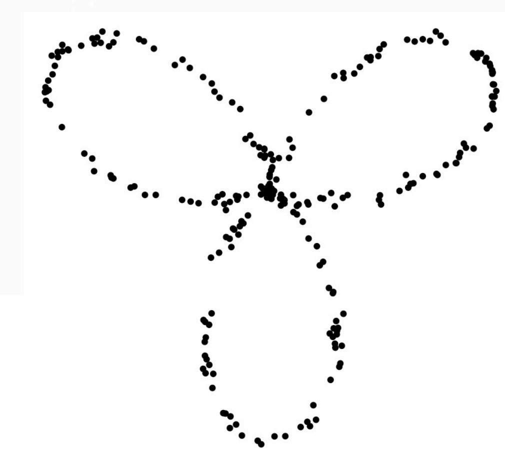


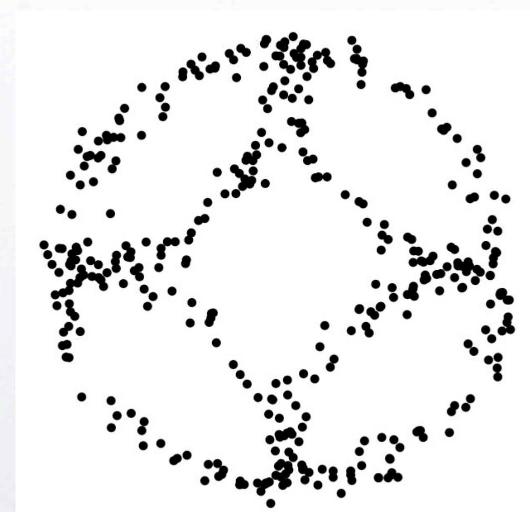


Point-cloud topology

- Topological structure in statistical data
  - density estimation and modality
  - approximation by simplicial complexes
- Assume data have been sampled from some unknown space
  - can we measure topological features of the hidden space?
  - can we assign confidence values to these measurements?
- What does "topology" mean for a cloud of data points?
  - persistent homology
  - spectral theory for point clouds







Betti numbers and homology groups

Betti numbers and homology groups

- Betti numbers and homology groups count and identify topological features
  - number of connected components, number of holes, etc.

Betti numbers and homology groups

- Betti numbers and homology groups count and identify topological features
  - number of connected components, number of holes, etc.
- The k-th Betti number of X is a non-negative integer $b_k(X)$
  - “b_k measures the k-dimensional connectivity of X”
  - “b_k counts the number of independent k-dimensional features of X”

Betti numbers and homology groups

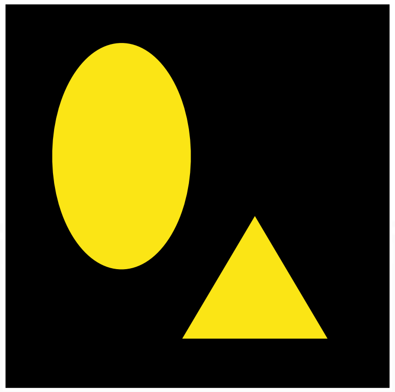
- Betti numbers and homology groups count and identify topological features
  - number of connected components, number of holes, etc.
- The k-th Betti number of X is a non-negative integer $b_k(X)$
  - "b_k measures the k-dimensional connectivity of X"
  - "b_k counts the number of independent k-dimensional features of X"
- The k-th homology group of X is a vector space $H_k(X)$
  - it is convenient use vector spaces over the scalar field {0,1}
  - each vector in $H_k(X)$ corresponds to a specific feature or combination of features

Betti numbers and homology groups

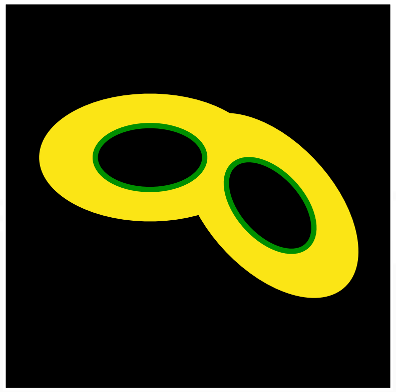
- Betti numbers and homology groups count and identify topological features
  - number of connected components, number of holes, etc.
- The k-th Betti number of X is a non-negative integer $b_k(X)$
  - "b_k measures the k-dimensional connectivity of X"
  - "b_k counts the number of independent k-dimensional features of X"
- The k-th homology group of X is a vector space $H_k(X)$
  - it is convenient use vector spaces over the scalar field {0,1}
  - each vector in $H_k(X)$ corresponds to a specific feature or combination of features
- Betti numbers and homology groups are related by $b_k(X) = \dim(H_k(X))$

2- and 3-dimensional examples

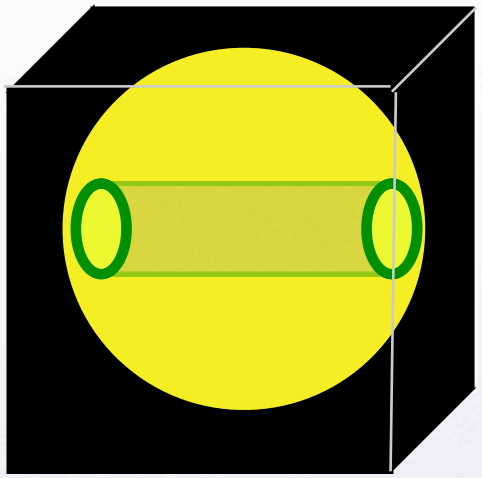
- For an object in 2-dimensional space
  - b_0 is the number of components
  - b_1 is the number of holes
- For an object in 3-dimensional space
  - b_0 is the number of components
  - b_1 is the number of tunnel or handles
  - b_2 is the number of voids
- (and so on, in higher dimensions)



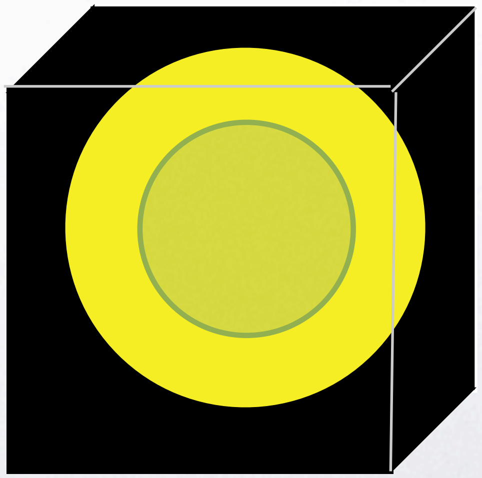
$b_0 = 2, b_1 = 0$



$b_0 = 1, b_1 = 2$



$b_0 = 1, b_1 = 1, b_2 = 0$



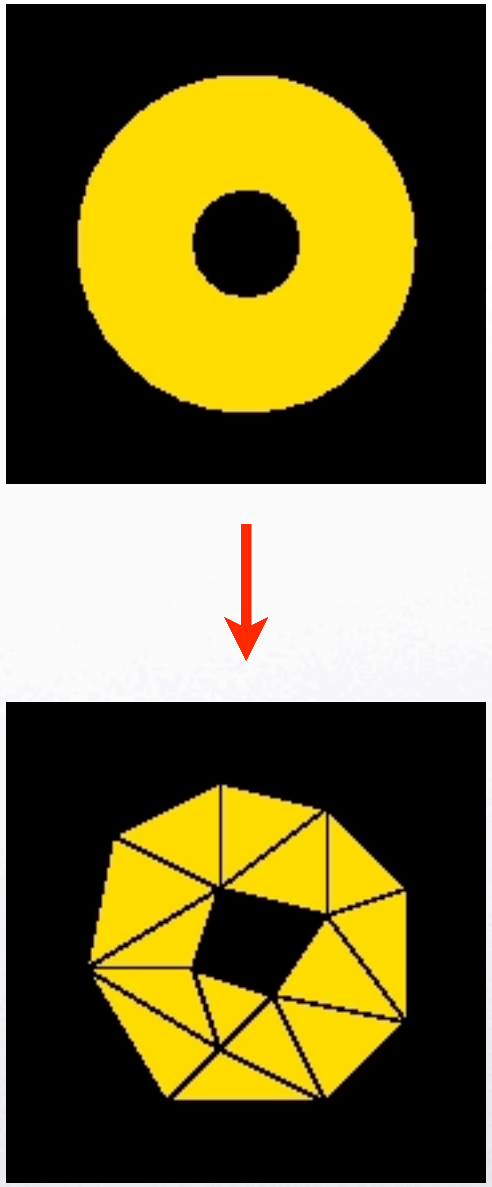
$b_0 = 1, b_1 = 0, b_2 = 1$

Calculating homology and Betti numbers

- Betti numbers and homology groups are defined for abstract topological spaces
  - this involves infinite-dimensional linear algebra
- A topological space can often be represented as finite simplicial complex
  - assembled from vertices, edges, triangles, tetrahedra, etc.
  - linear algebra becomes finite dimensional
  - “simplicial homology”

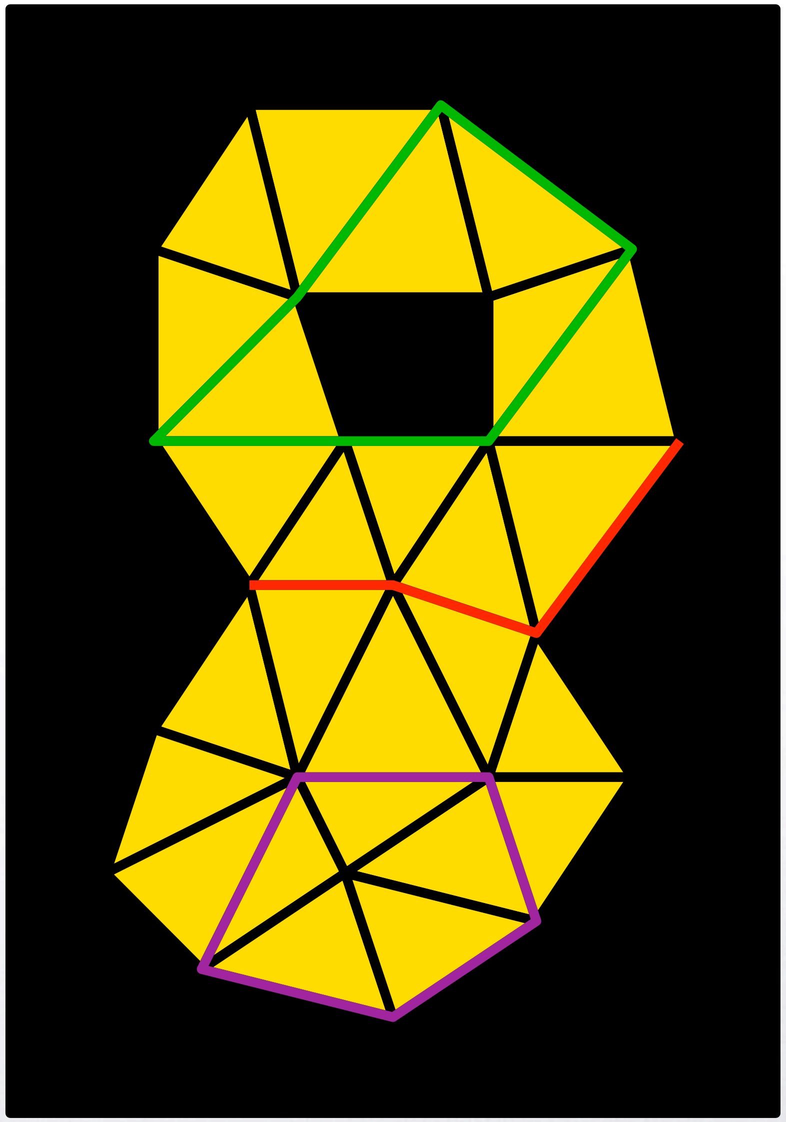
Calculating homology and Betti numbers

- Betti numbers and homology groups are defined for abstract topological spaces
  - this involves infinite-dimensional linear algebra
- A topological space can often be represented as finite simplicial complex
  - assembled from vertices, edges, triangles, tetrahedra, etc.
  - linear algebra becomes finite dimensional
  - “simplicial homology”



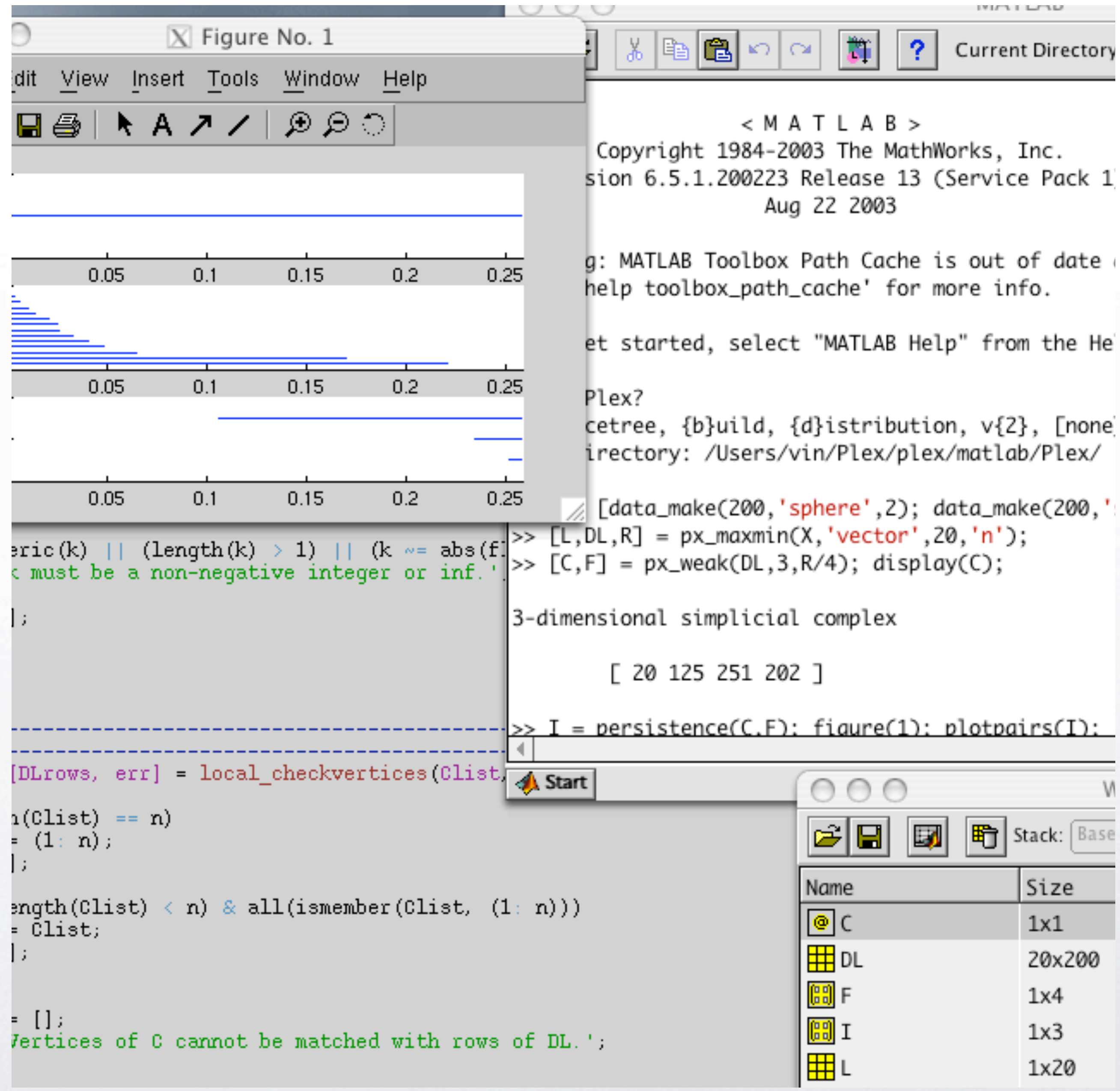
Simplicial homology

- Simplicial complex X
- n_0 vertices, n_1 edges, n_2 triangles, ...
- Define vector spaces C_0, C_1, C_2, ...
  - $C_i \leftrightarrow$ {subsets of the set of all i-simplices}
- Define linear maps ∂_i: $C_{i+1} \to C_i$
  - each (i+1)-simplex maps to its set of bounding i-simplices
  - count i-simplices modulo 2
  - the boundary of a boundary is empty: $\partial^2 = 0$
- Define $H_i(X) = \mathrm{Ker}(\partial_i) / \mathrm{Im}(\partial_{i-1})$
- Define $b_i(X) = \dim(H_i(X))$



Plex: algebraic topology in MATLAB

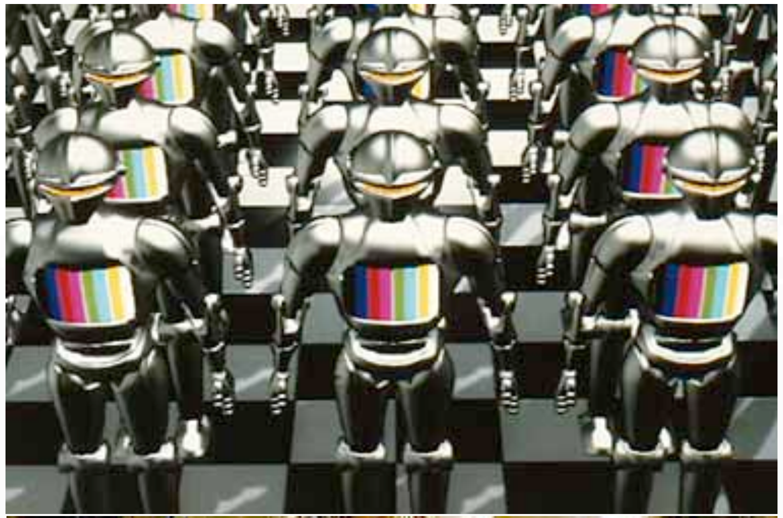
- Plex is a library of C++ and MATLAB routines for applied algebraic topology
- MATLAB front-end allows for easy high-level scripting
- Version 1 (VdS, 2000-3)
- Version 2 (Pat Perry & VdS, 2005)
  - core library written in C++
  - “metric data” toolbox
  - includes persistent homology library of Afra Zomorodian and Lutz Kettner
- Version 3 (Harlan Sexton, 2007)
  - Java-based, for portability

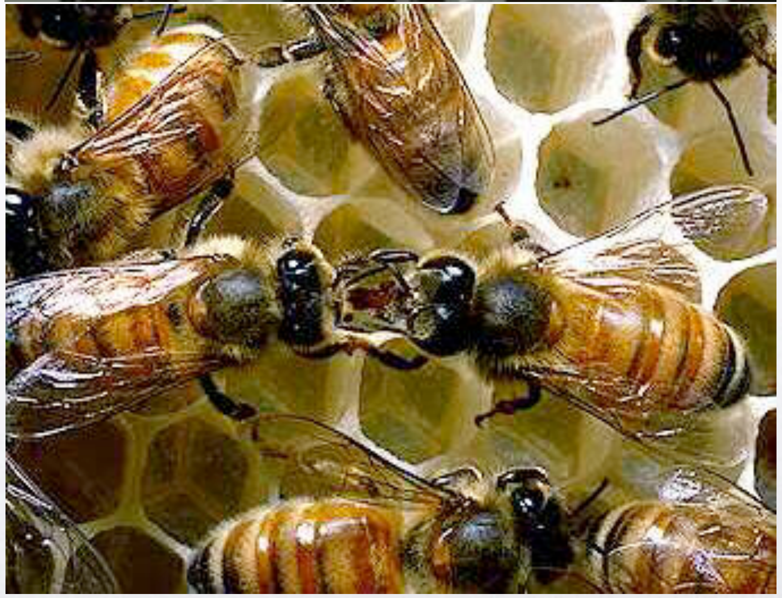


Sensor networks

Sensor networks

- We deploy a large number of independent robotic agents
  - dozens, hundreds, thousands, ...
- Each robot has limited physical and computational capabilities
  - optical/aural sensing
  - locomotion
  - communication with nearby robots
- Attempt to solve global problems using local algorithms
  - each robot has simple behaviour rules
  - "whole is greater than sum of parts"



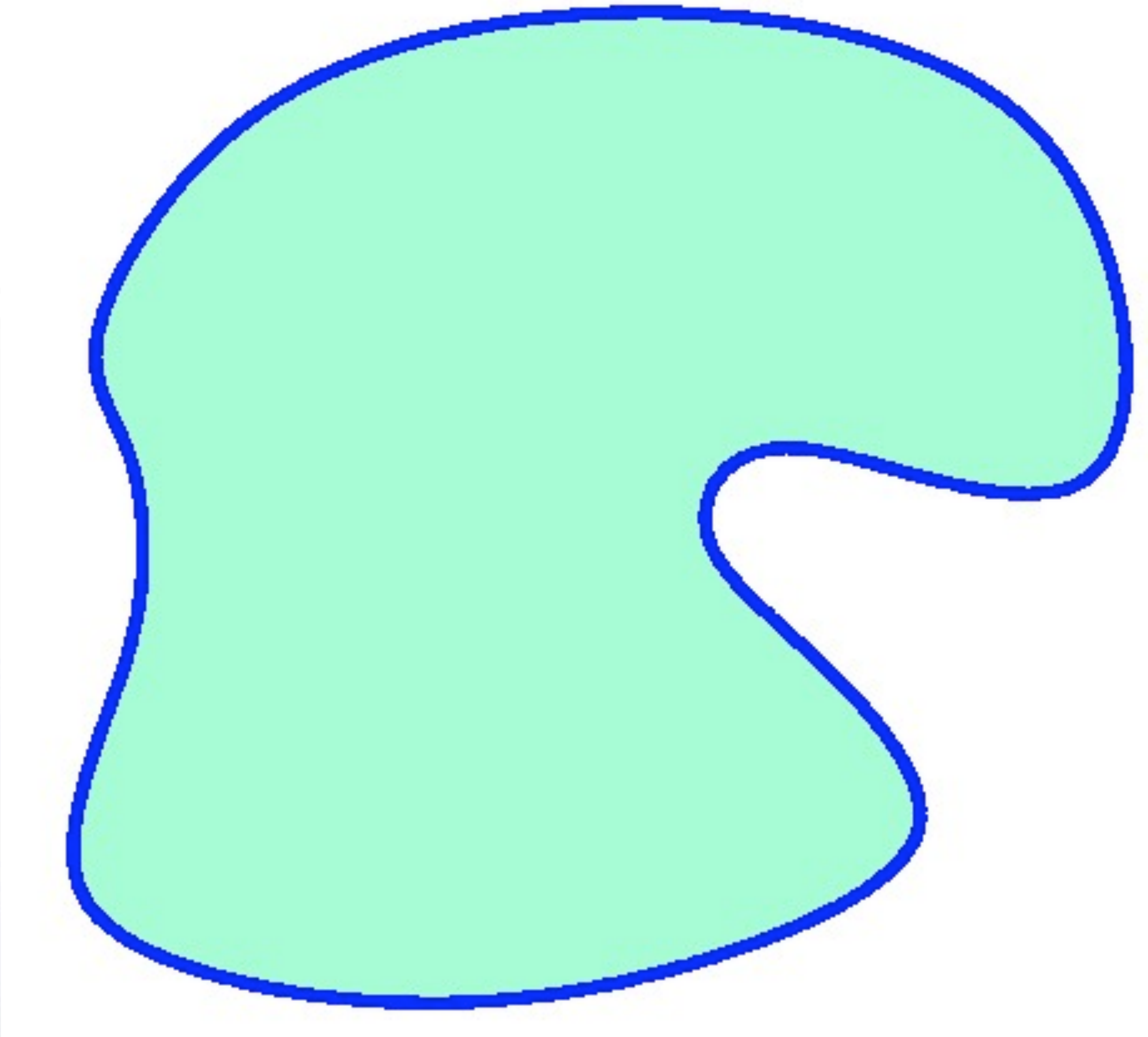


The coverage problem

- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance

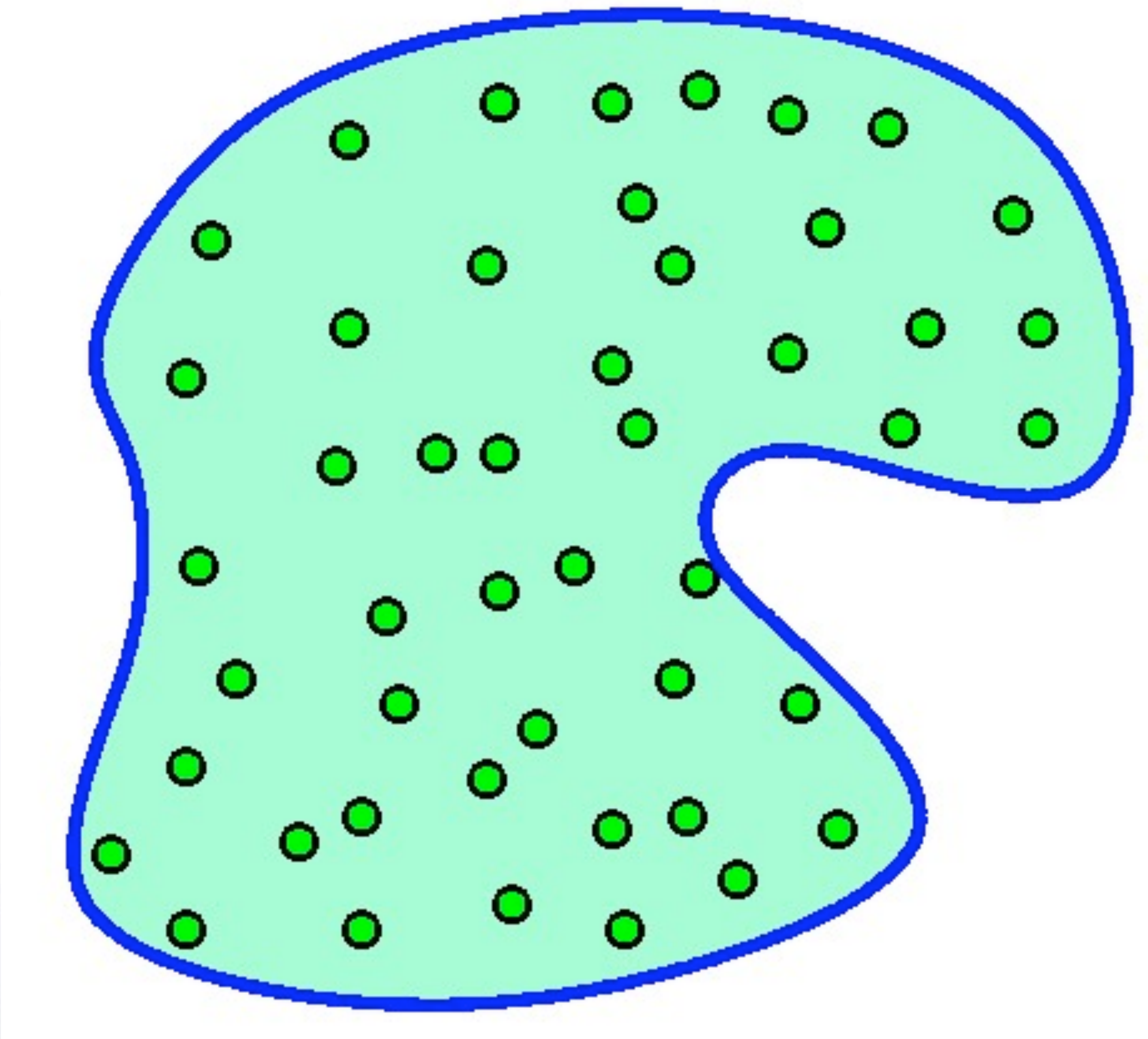
The coverage problem

- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance



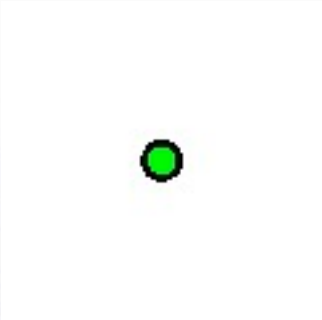
The coverage problem

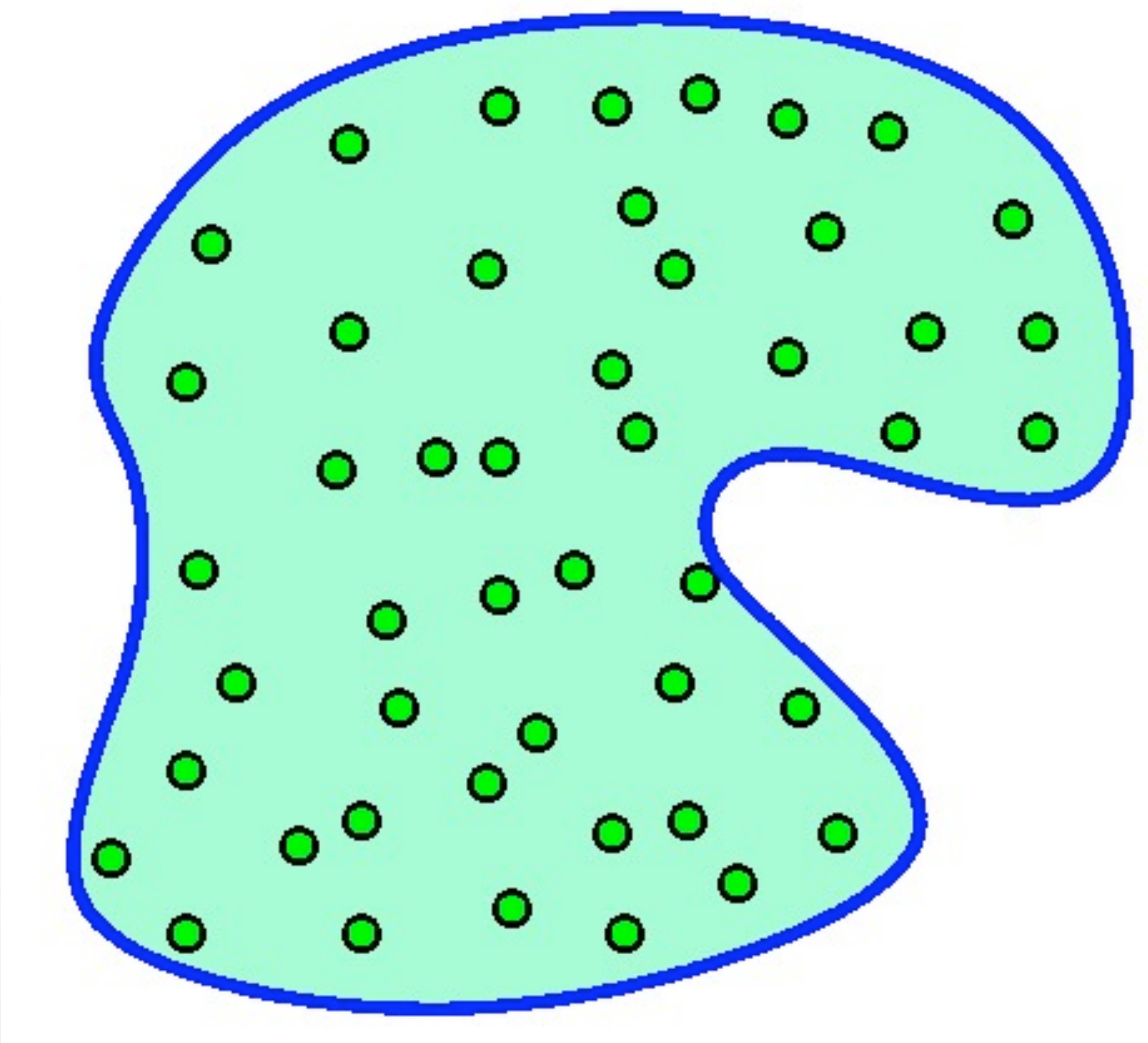
- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance



The coverage problem

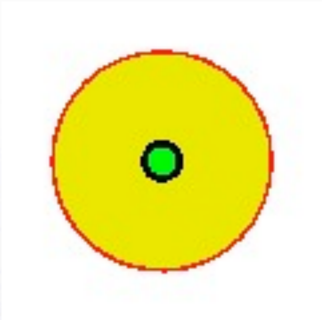
- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance

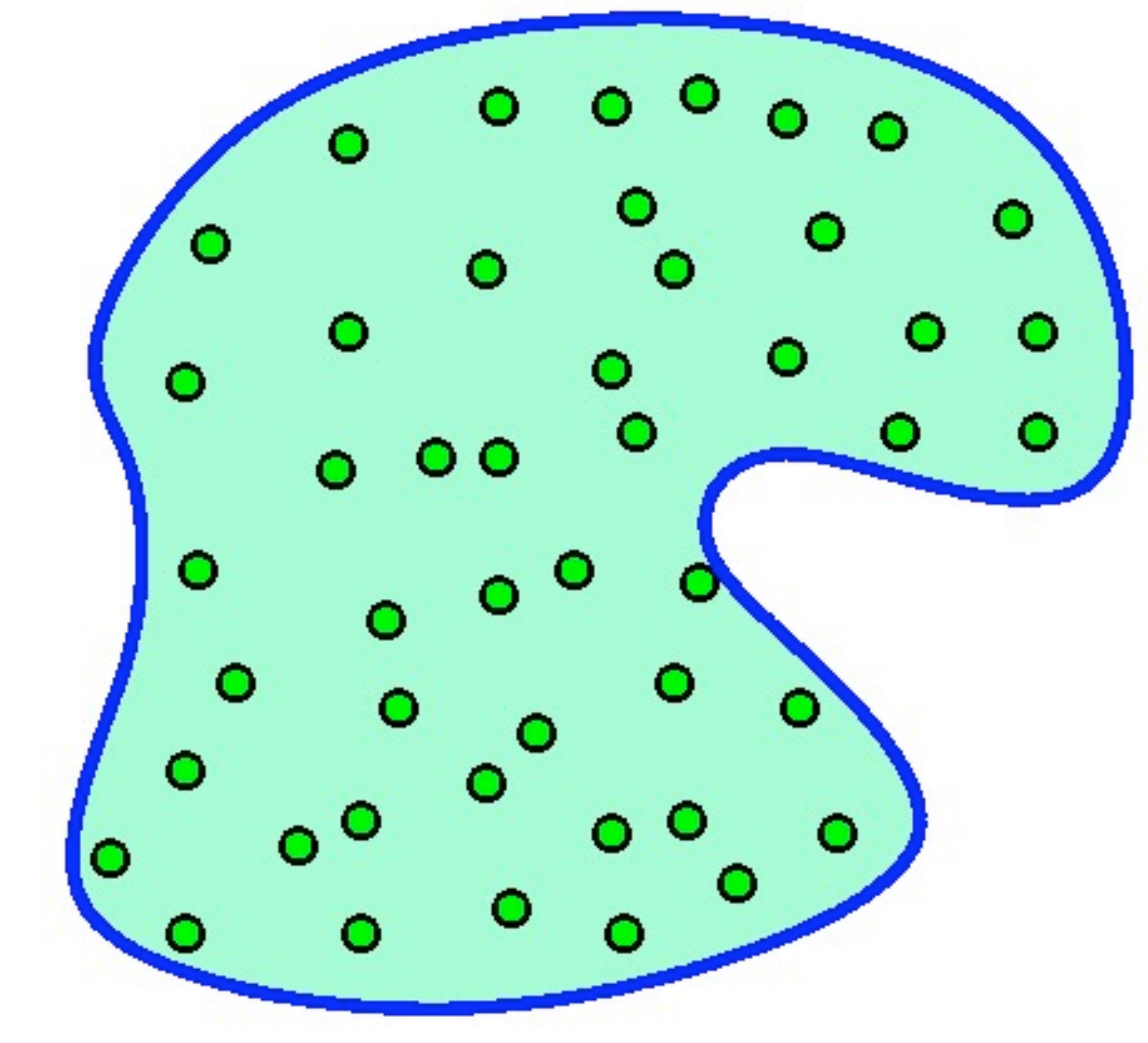




The coverage problem

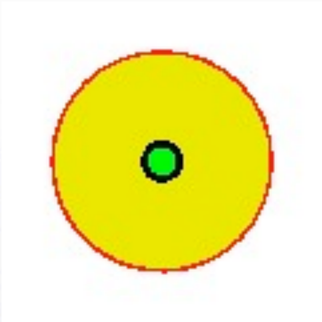
- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance

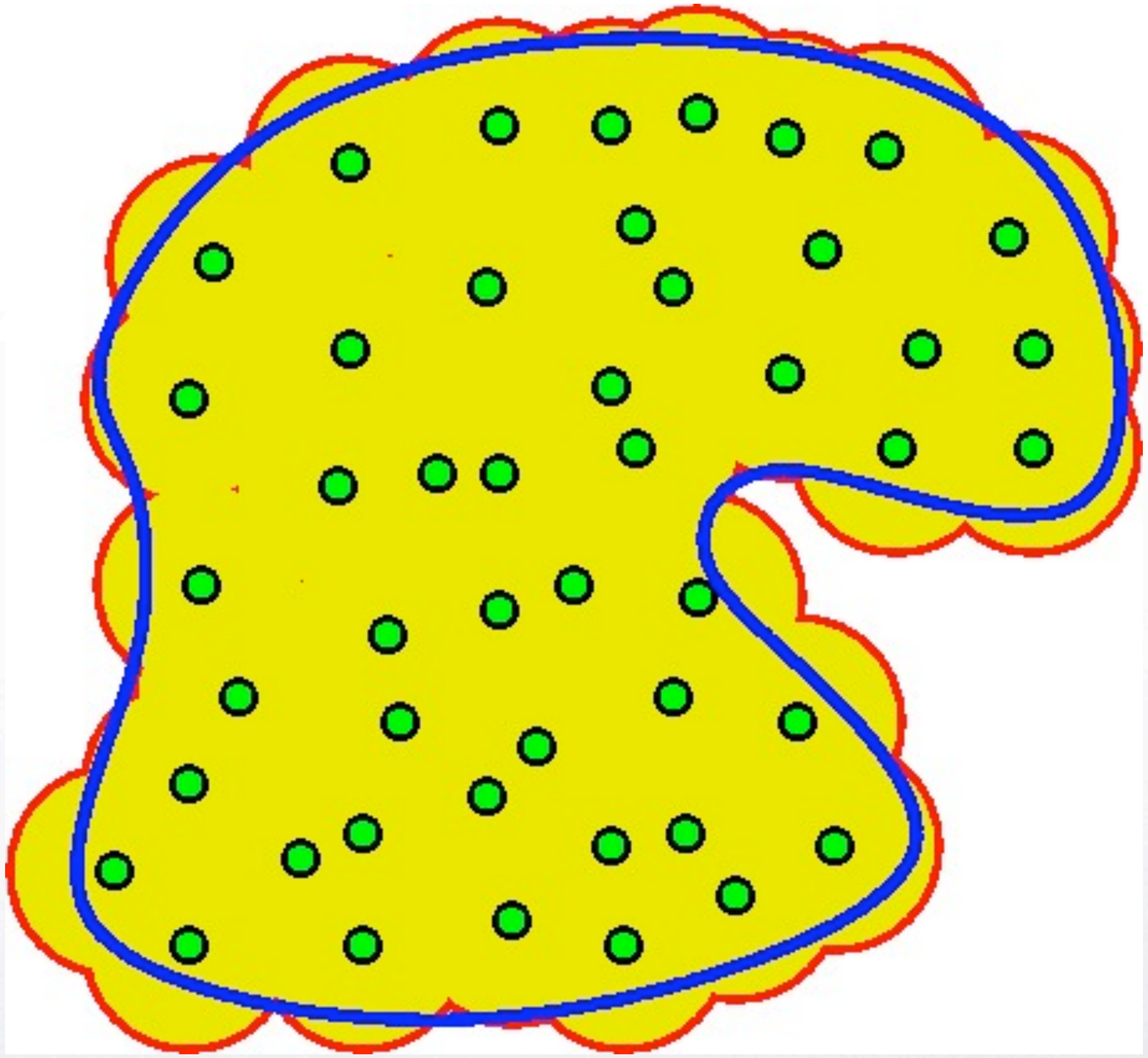




The coverage problem

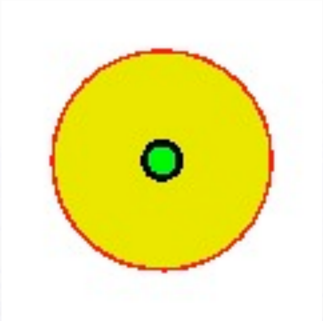
- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance

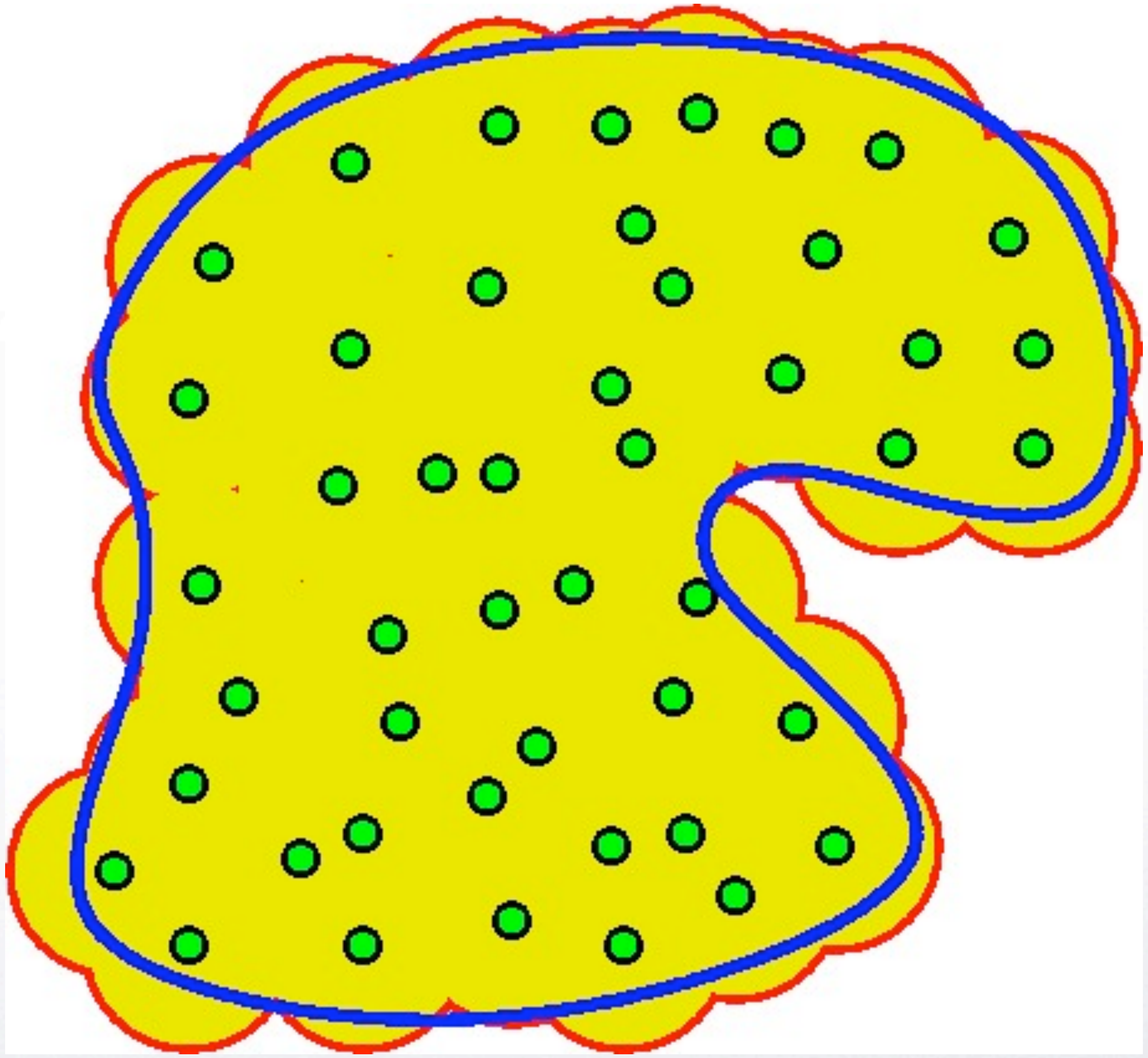




The coverage problem

- 2D domain bounded by fence
- Robots populate the domain
- Each robot has a coverage area
  - signal transmission
  - surveillance





Is the entire domain covered?

Attacking the coverage problem

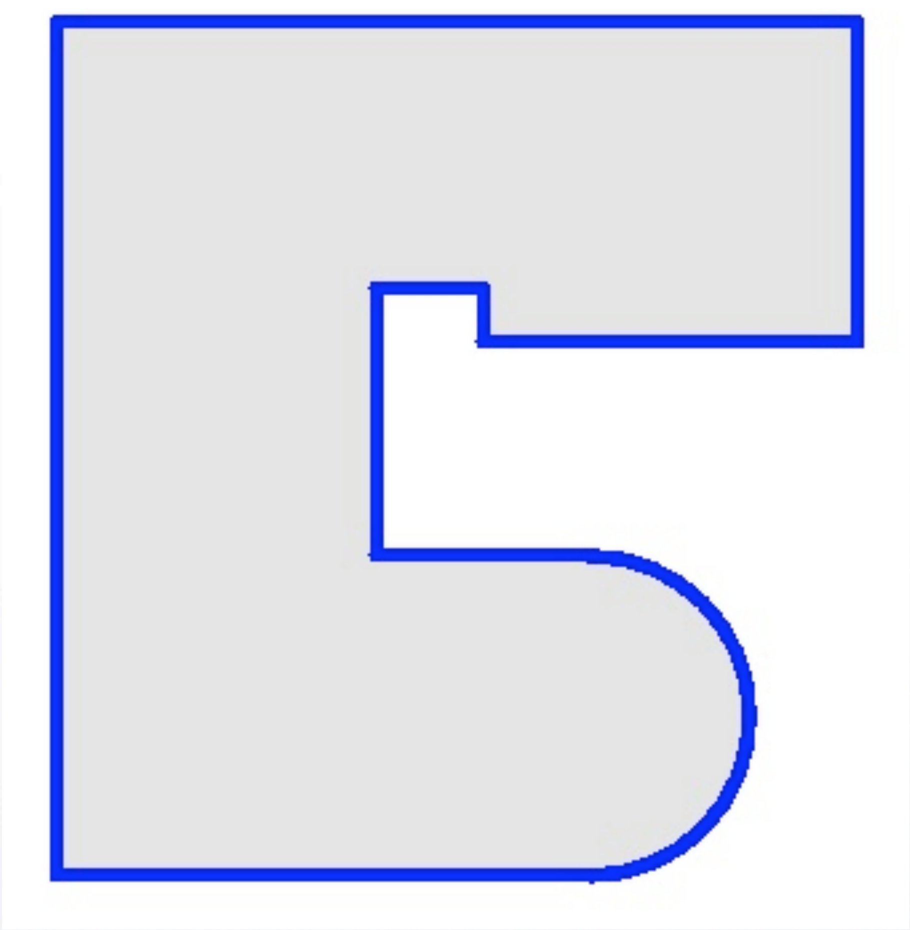
What could we use to solve the problem?

- Exact knowledge of domain shape
  - "exploring the known"
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - "mission control"

Attacking the coverage problem

What could we use to solve the problem?

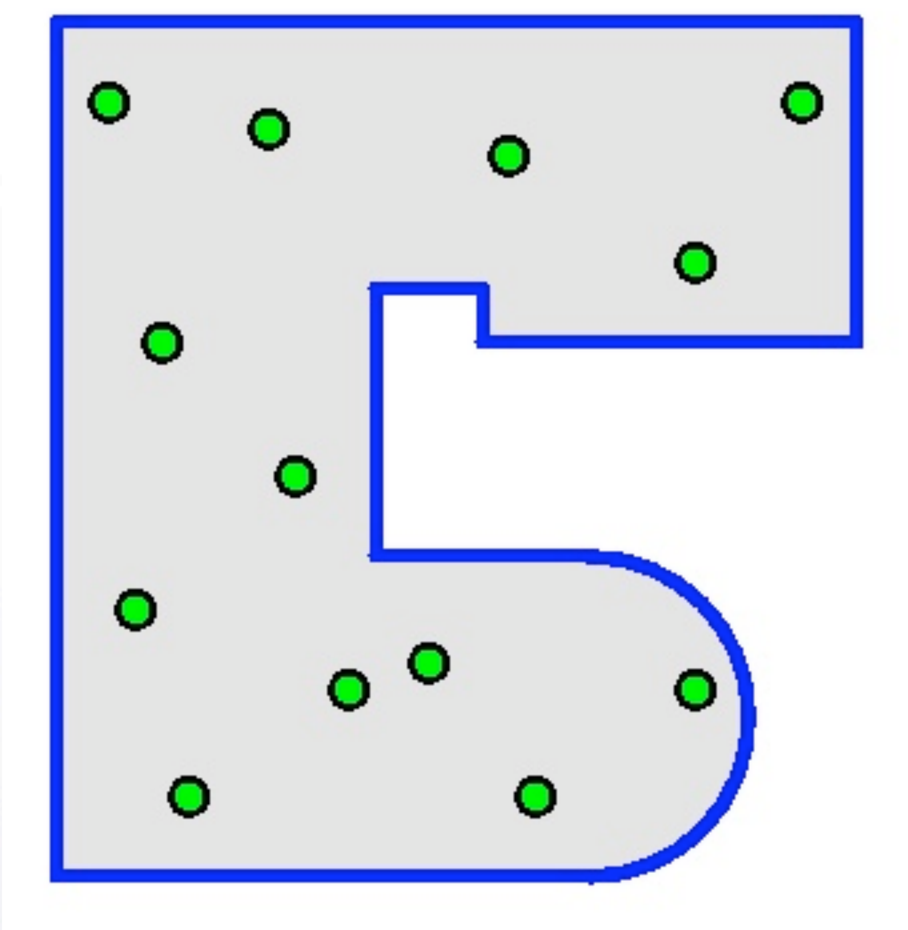
- Exact knowledge of domain shape
  - "exploring the known"
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - "mission control"



Attacking the coverage problem

What could we use to solve the problem?

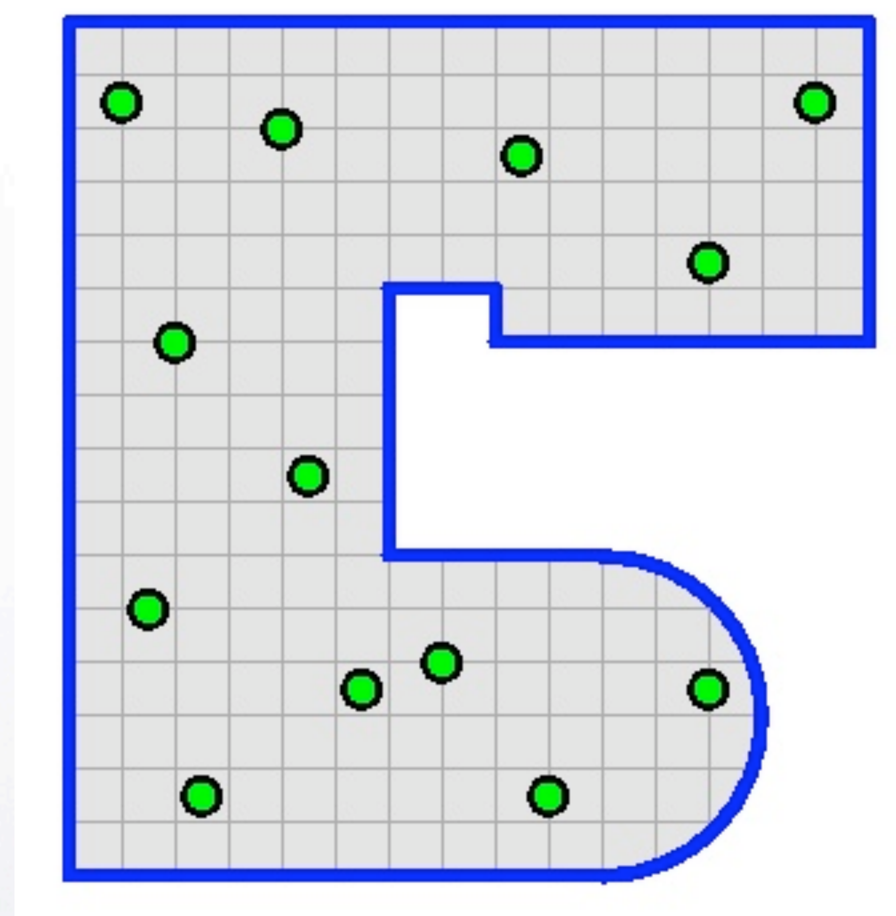
- Exact knowledge of domain shape
  - "exploring the known"
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - "mission control"



Attacking the coverage problem

What could we use to solve the problem?

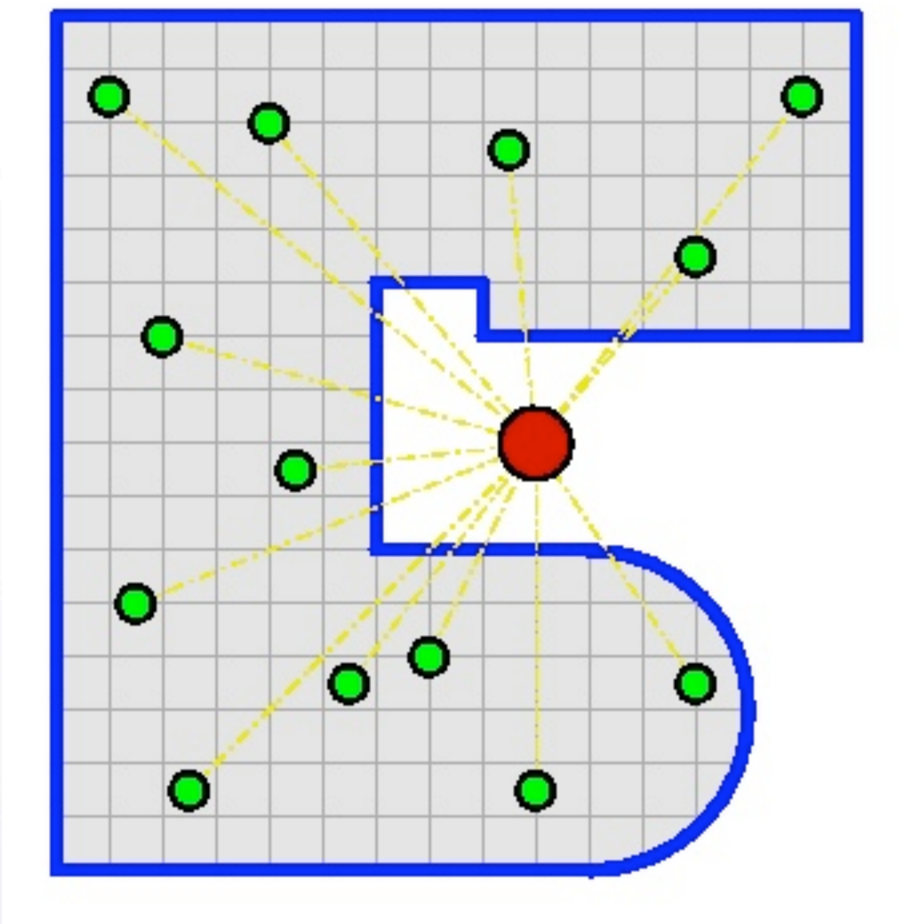
- Exact knowledge of domain shape
  - “exploring the known”
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - “mission control”



Attacking the coverage problem

What could we use to solve the problem?

- Exact knowledge of domain shape
  - “exploring the known”
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - “mission control”



Attacking the coverage problem

What could we use to solve the problem?

- Exact knowledge of domain shape
  - "exploring the known"
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - "mission control"

Attacking the coverage problem *using topology*

What could we use to solve the problem?

- Exact knowledge of domain shape
  - "exploring the known"
- Exact knowledge of robot positions
  - e.g. using GPS systems
- Centralised information gathering and computation
  - "mission control"

Attacking the coverage problem *using topology*

What could we use to solve the problem?

- ~~Exact knowledge of domain shape~~
  - "exploring the ~~known~~ *unknown*"
- ~~Exact knowledge of robot positions~~
  - ~~e.g. using GPS systems~~
- Centralised information gathering and computation
  - "mission control"

unknown domain shape
- with mild constraints

crude proximity information
- identify nearby robots and fence

Theorem (VdS, Ghrist, Muhammad 2005)

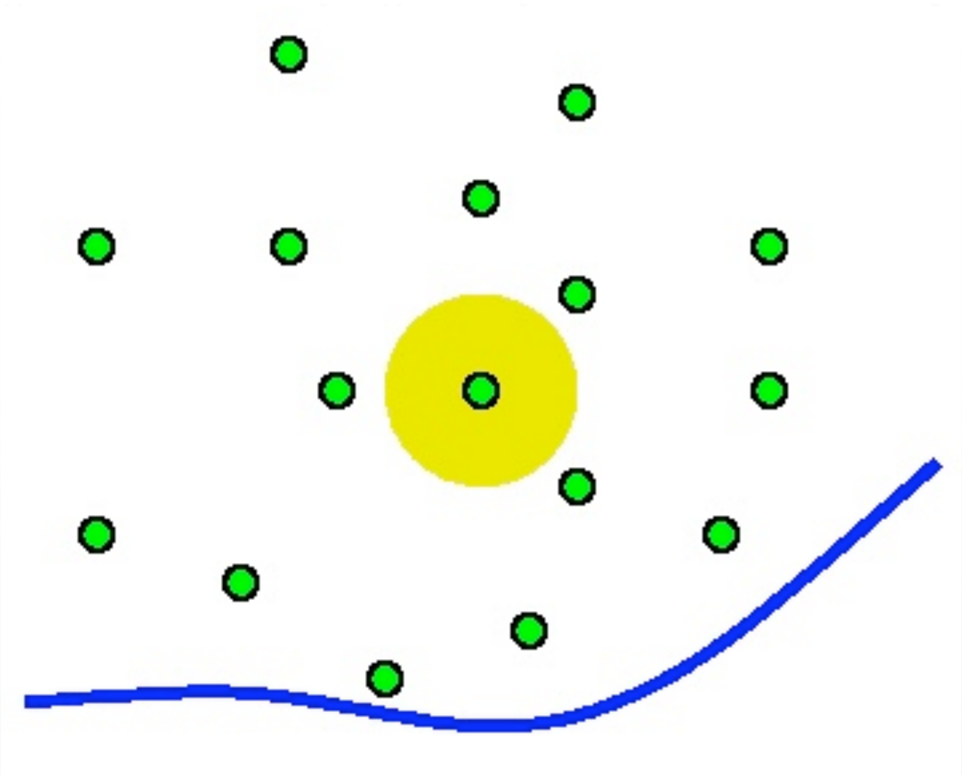
- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives

Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives

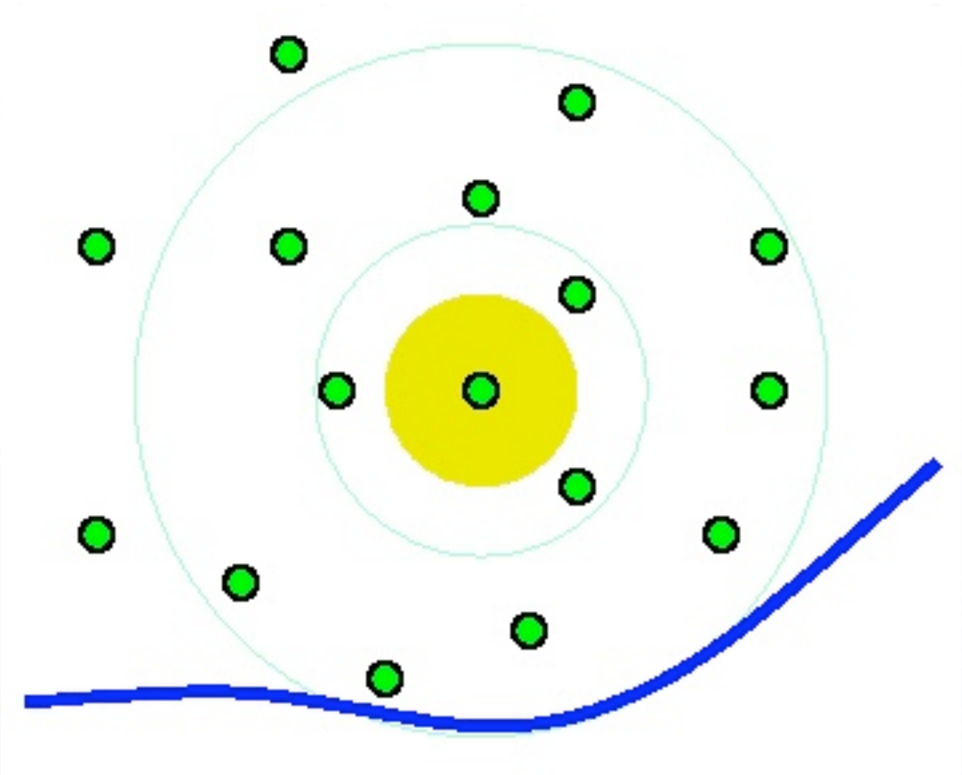
Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives



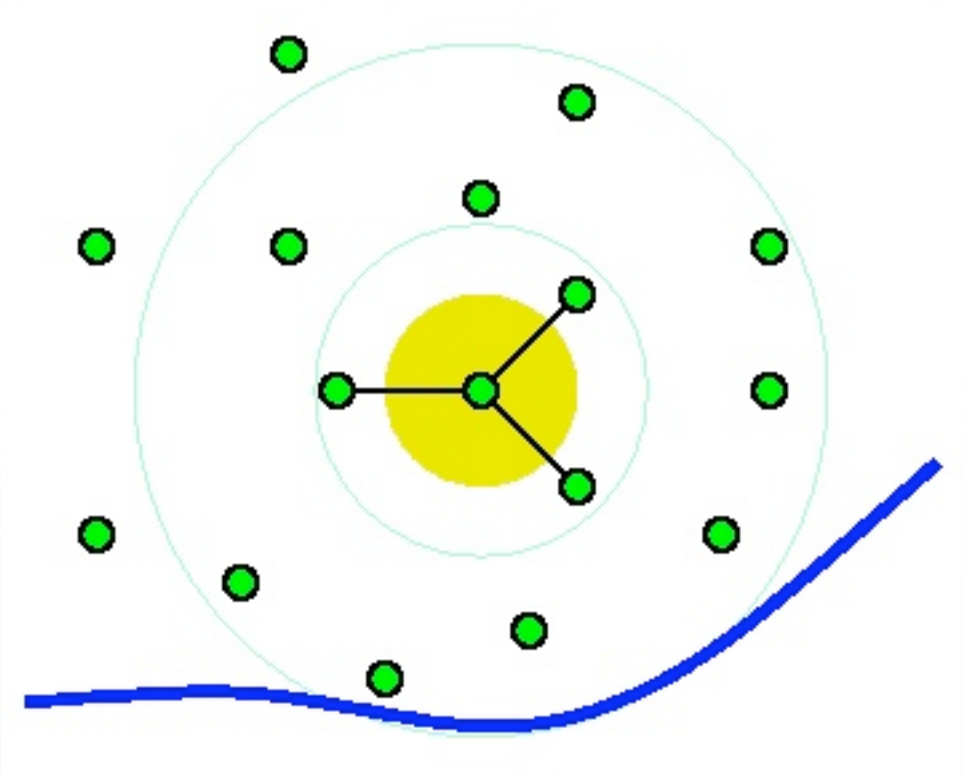
Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives



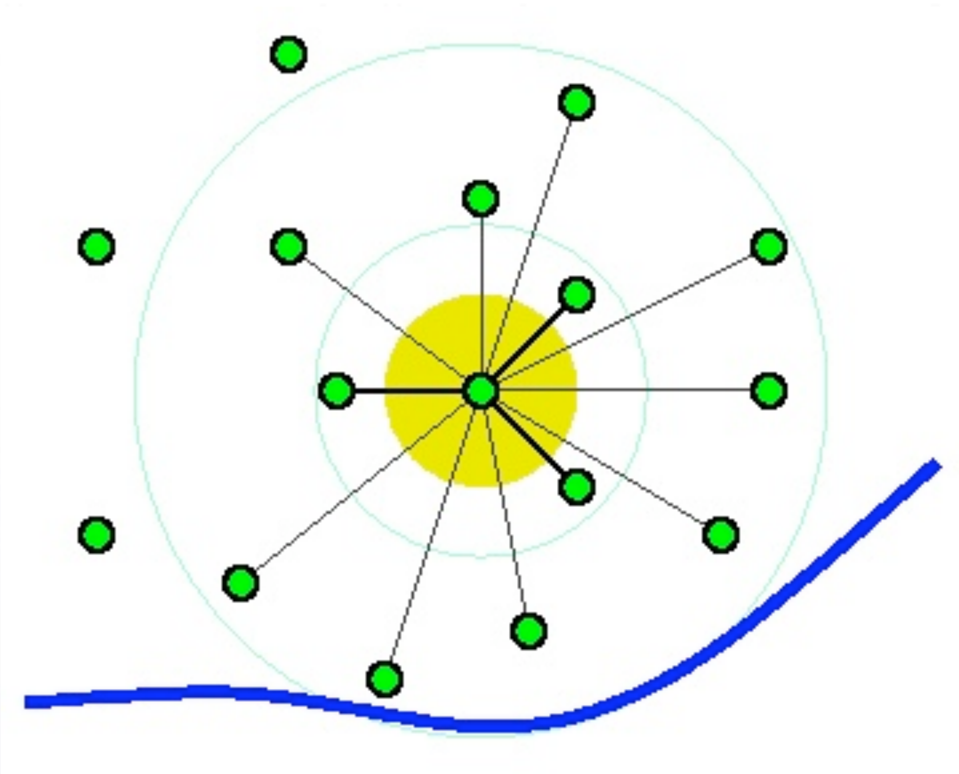
Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives



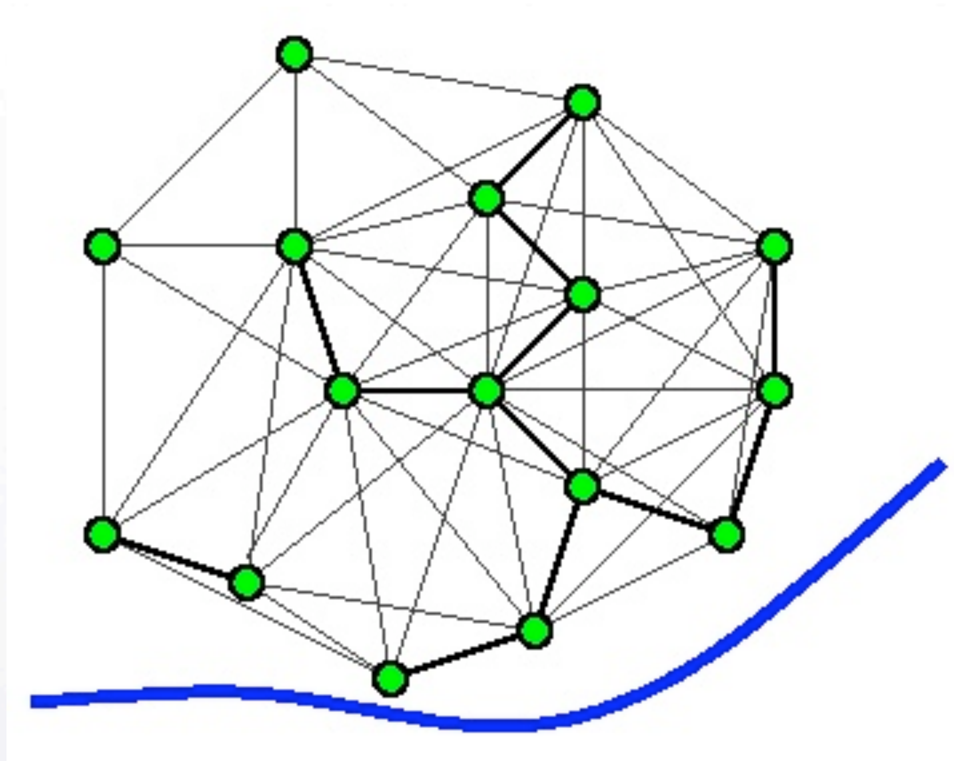
Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives



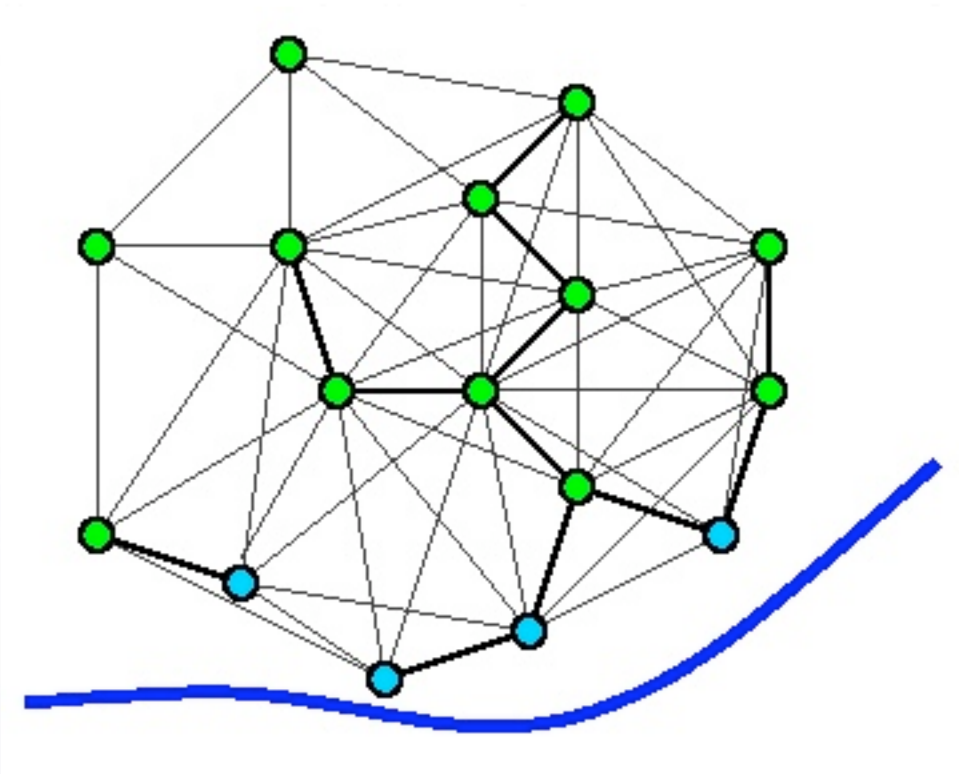
Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives

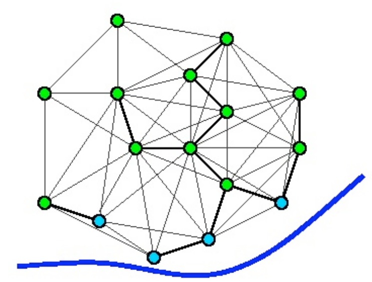


Theorem (VdS, Ghrist, Muhammad 2005)

- Assumptions
  - The coverage area of each robot is a circular disk of radius r_c
  - Each robot can identify all robots which are near it (distance $\leq r_s$)
  - Each robot can identify all robots which are at midrange (distance $\leq r_w$)
  - Each robot knows if it is close to the fence (distance $\leq r_f$)
  - $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - The domain is not "pinched"
  - The fence is "not too wiggly"
- Conclusion
  - There is a test for global coverage which gives no false positives, and "not too many" false negatives

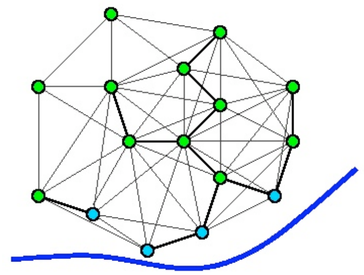


2-dimensional coverage: first try



- Construct a simplicial complex R ("Rips complex")
  - a vertex for each robot
  - an edge whenever two robots are separated by distance at most $r_c\sqrt{3}$
  - all triangles for which all the edges already belong to R
- Construct a subcomplex F ("Fence complex")
  - a vertex for each robot within distance r_f of the boundary
  - all edges, triangles for which all the vertices already belong to F
- Are the following statements equivalent?
  - coverage is achieved in the interior of the domain
  - there is a 2-chain in R with boundary in F which does not retract into F
  - in relative homology $H_2(R,F) \neq 0$

2-dimensional coverage: first try



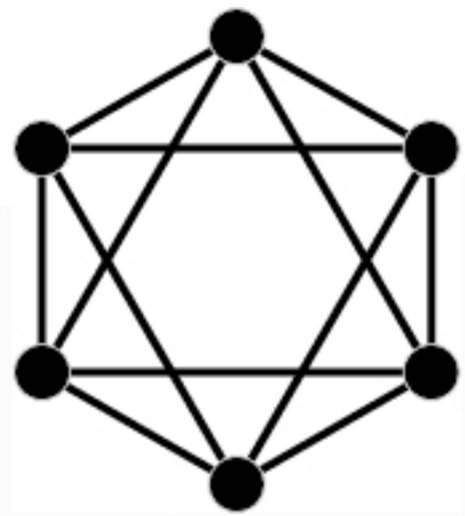
- Construct a simplicial complex R ("Rips complex")
  - a vertex for each robot
  - an edge whenever two robots are separated by distance at most $r_c\sqrt{3}$
  - all triangles for which all the edges already belong to R
- Construct a subcomplex F ("Fence complex")
  - a vertex for each robot within distance r_f of the boundary
  - all edges, triangles for which all the vertices already belong to F
- Are the following statements equivalent?
  - coverage is achieved in the interior of the domain
  - there is a 2-chain in R with boundary in F which does not retract into F
  - in relative homology $H_2(R,F) \neq 0$

The √3 Lemma

- Definition An r_s-triangle is a triangle where all three sides have length ≤ r_s.
- Lemma If three robots lie at the vertices of an r_s-triangle, and if $r_c \geq r_s/\sqrt{3}$ then the three coverage disks of radius r_c meet and cover the entire triangle.
  - worst case: equilateral triangle
- More generally, in d dimensions: if all the edges of a d-simplex have length at most $r_c\sqrt{(2+2/d)}$, then the $d+1$ balls of radius r_c meet and cover the entire simplex.
  - worst case: regular simplex
- If we can find a set of r_s-triangles covering the domain with a robot at each vertex, then by Lemma we have coverage.

False 2-cycles

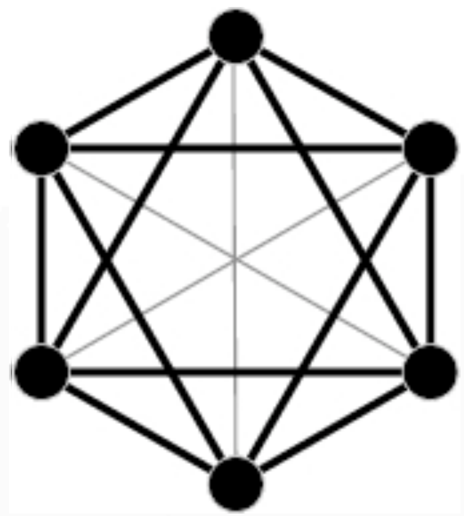
It is not enough to find a nonzero vector in $H_2(R,F)$



This closed system of 8 triangles cannot be collapsed using tetrahedra...

False 2-cycles

It is not enough to find a nonzero vector in $H_2(R,F)$

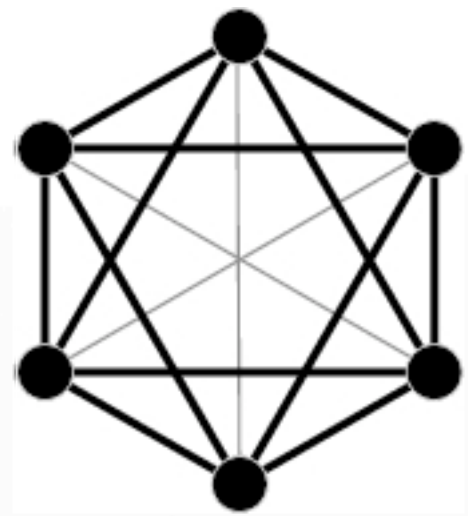


This closed system of 8 triangles cannot be collapsed using tetrahedra...

...unless you allow slightly larger tetrahedra.

False 2-cycles

It is not enough to find a nonzero vector in $H_2(R,F)$



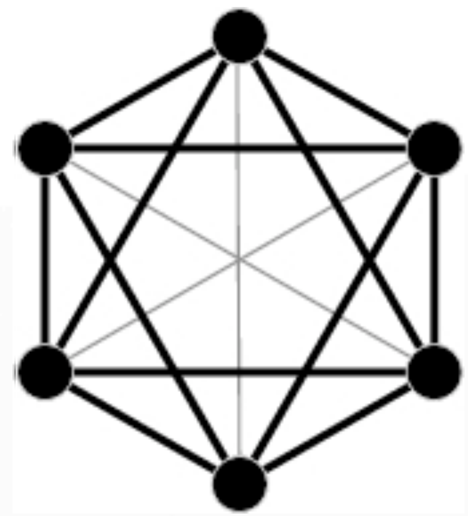
This closed system of 8 triangles cannot be collapsed using tetrahedra...

...unless you allow slightly larger tetrahedra.

Theorem Any closed system of r_s-triangles in the plane can be collapsed as the boundary of a set of r_w-tetrahedra, provided that $r_w \geq 2r_s/\sqrt{3}$.

False 2-cycles

It is not enough to find a nonzero vector in $H_2(R,F)$



This closed system of 8 triangles cannot be collapsed using tetrahedra...
...unless you allow slightly larger tetrahedra.

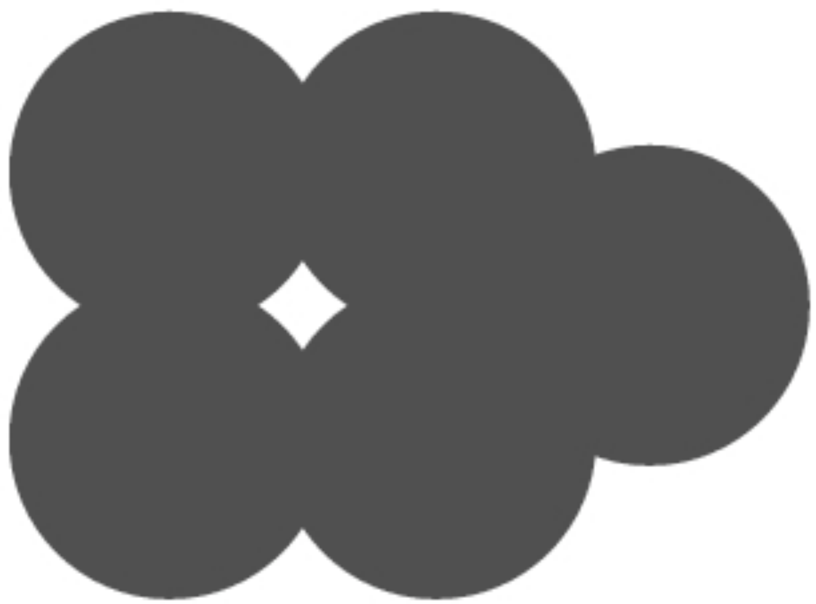
Theorem Any closed system of r_s-triangles in the plane can be collapsed as the boundary of a set of r_w-tetrahedra, provided that $r_w \geq 2r_s/\sqrt{3}$.

Caution Worse things can happen when the fence is involved.

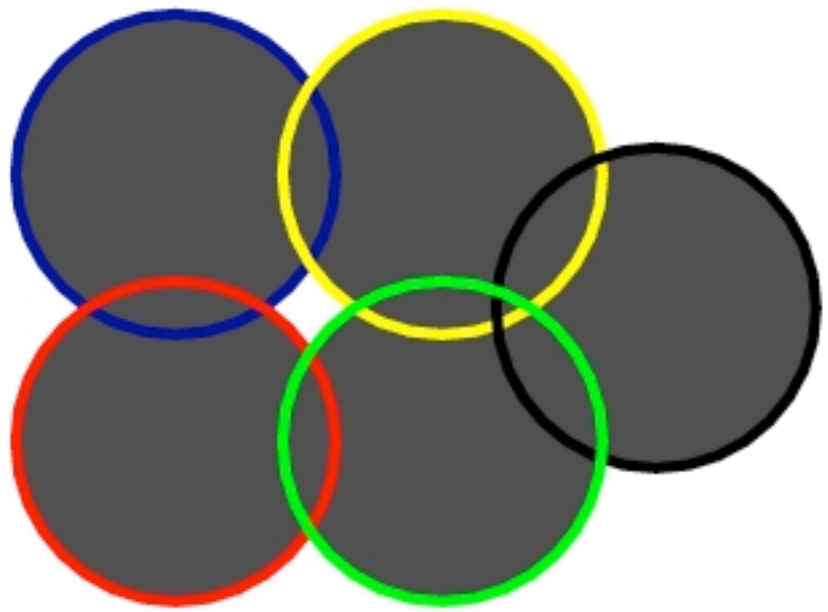
2-dimensional coverage: one possible answer

- Set three different radii r_s, r_w, r_f in addition to coverage radius r_c
  - require inequalities $r_c \geq r_s\sqrt{(1/3)}$ and $r_w \geq r_s\sqrt{(13/3)}$
  - geometry of domain is subject to constraints governed by r_s, r_w, r_f
- Construct simplicial complexes $R_s \subseteq R_w$
  - a vertex for each robot
  - an edge whenever two robots are separated by distance at most r_s, r_w
  - all triangles, tetrahedra for which all the edges already belong to R_s, R_w
- Construct subcomplexes F_s, F_w of R_s, R_w
  - a vertex for each robot within distance r_f of the boundary
  - all edges, triangles, tetrahedra for which all the vertices already belong to F_s, F_w
- Under the given assumptions
  - if the relative homology map $H_2(R_s,F_s) \to H_2(R_w,F_w)$ is not zero
  - then coverage is achieved (except possibly at points close to the boundary)

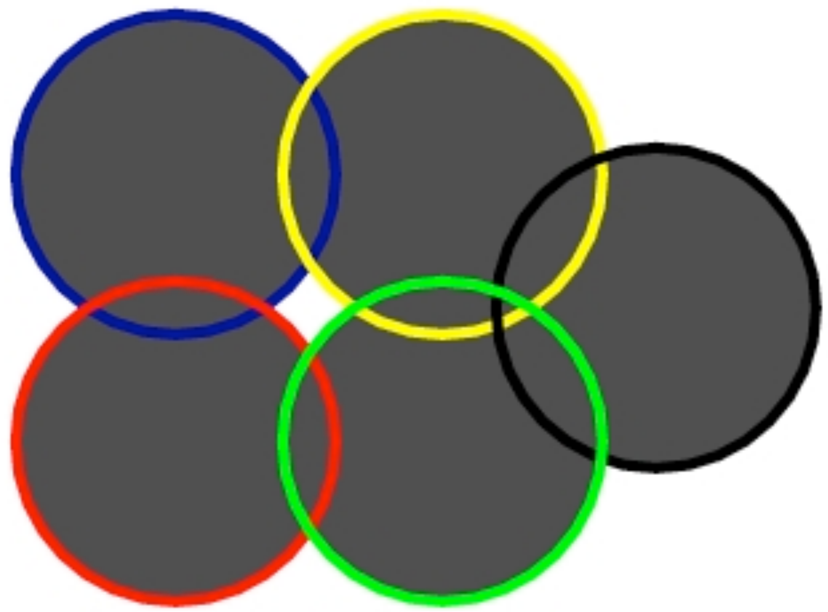
The nerve complex: a better approach?



The nerve complex: a better approach?

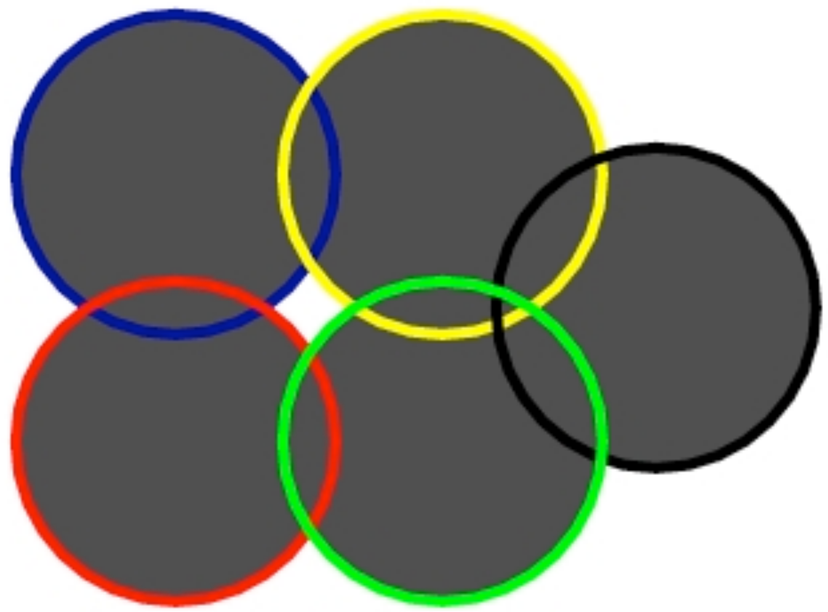


The nerve complex: a better approach?



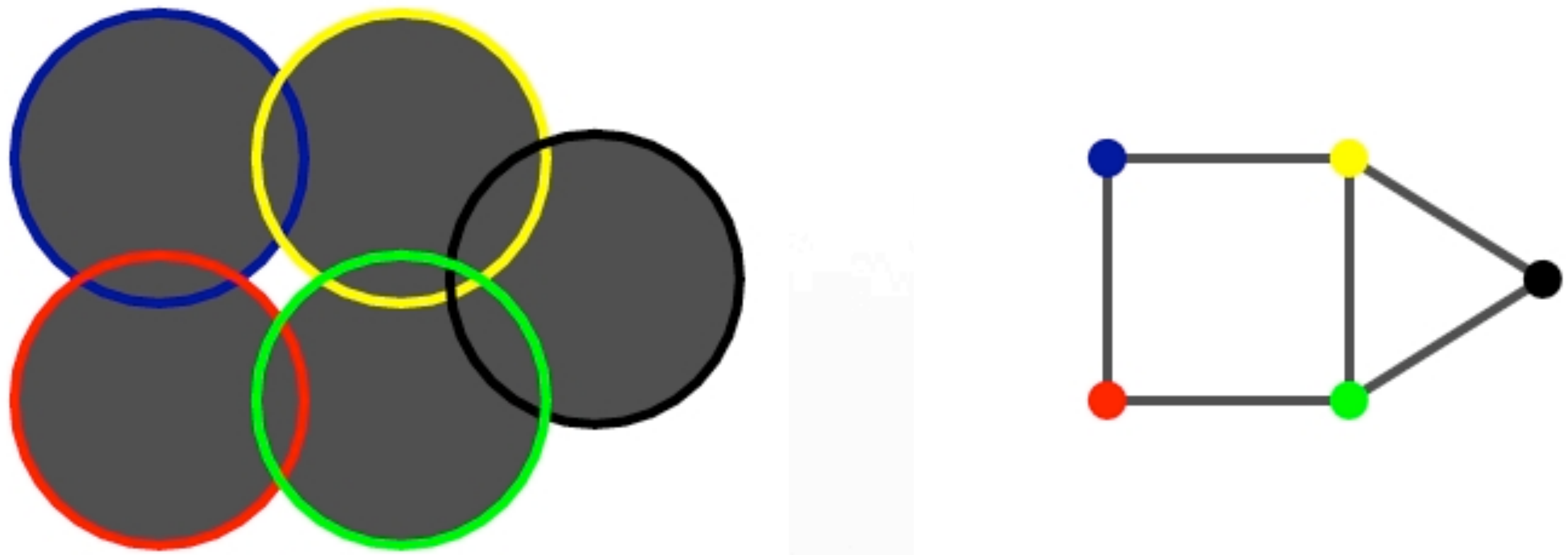
- For a union of disks in the plane we define a "nerve complex"
  - a vertex for every disk
  - an edge whenever two disks intersect
  - a triangle whenever three disks intersect
- ✔ The nerve complex has the same topology as the original union of disks
- ✘ We need precise inter-robot distances to calculate the nerve complex

The nerve complex: a better approach?



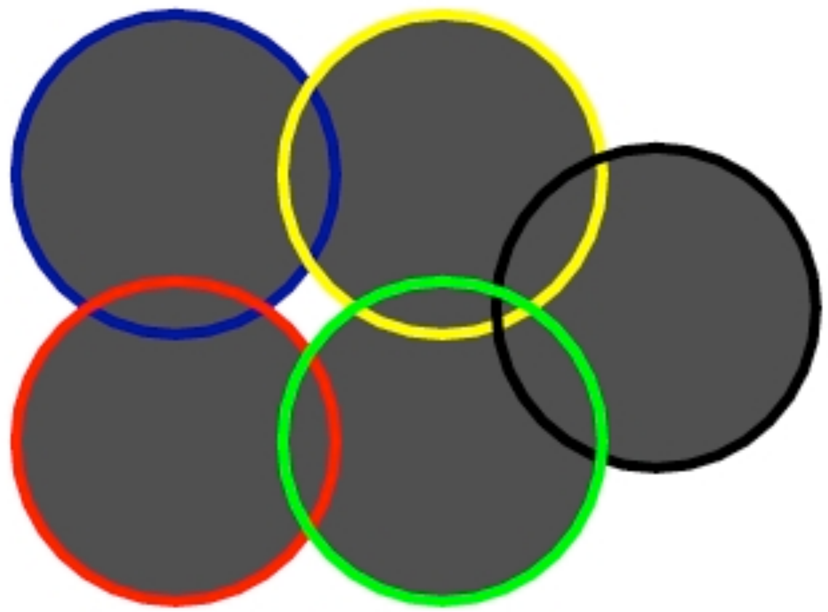
- For a union of disks in the plane we define a "nerve complex"
  - a vertex for every disk
  - an edge whenever two disks intersect
  - a triangle whenever three disks intersect
- ✔ The nerve complex has the same topology as the original union of disks
- ✘ We need precise inter-robot distances to calculate the nerve complex

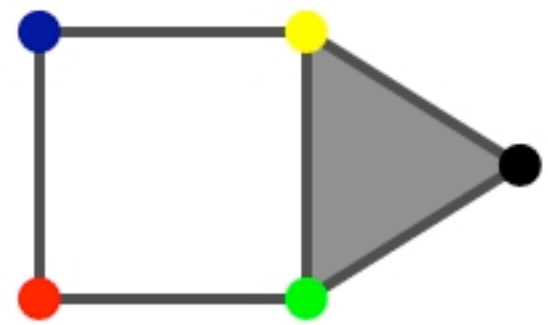
The nerve complex: a better approach?



- For a union of disks in the plane we define a "nerve complex"
  - a vertex for every disk
  - an edge whenever two disks intersect
  - a triangle whenever three disks intersect
- ✔ The nerve complex has the same topology as the original union of disks
- ✘ We need precise inter-robot distances to calculate the nerve complex

The nerve complex: a better approach?

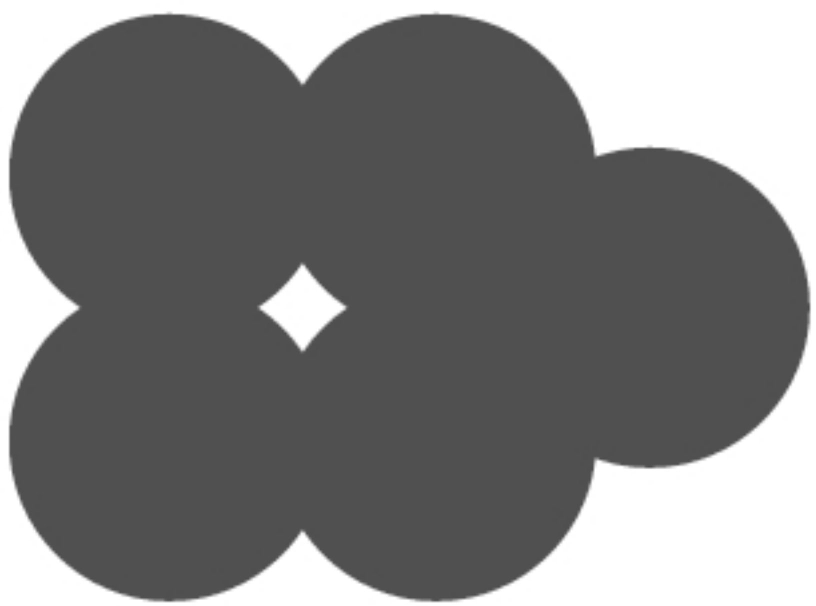


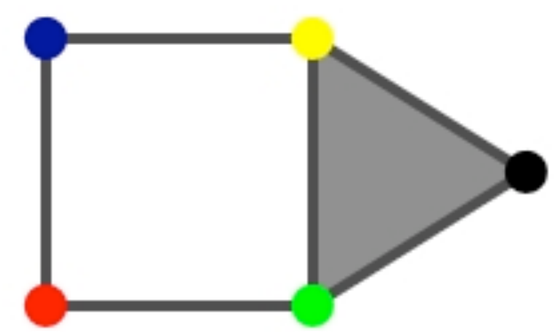


- For a union of disks in the plane we define a "nerve complex"
  - a vertex for every disk
  - an edge whenever two disks intersect
  - a triangle whenever three disks intersect
- ✔ The nerve complex has the same topology as the original union of disks
- ✘ We need precise inter-robot distances to calculate the nerve complex

Vin de Silva
http://pages.pomona.edu/~vds04747/

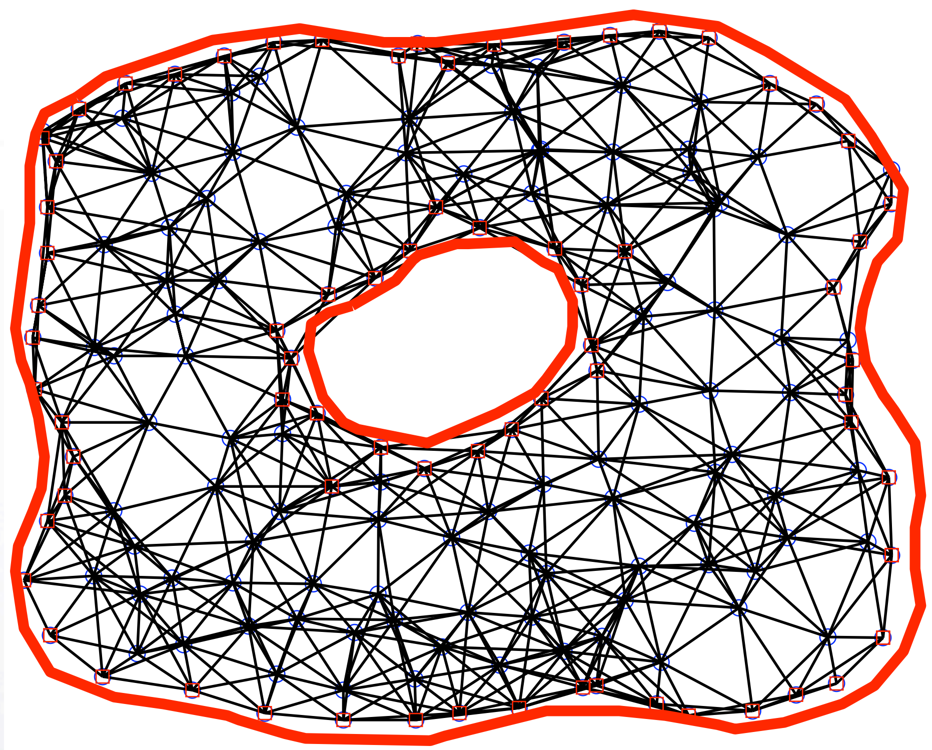
The nerve complex: a better approach?

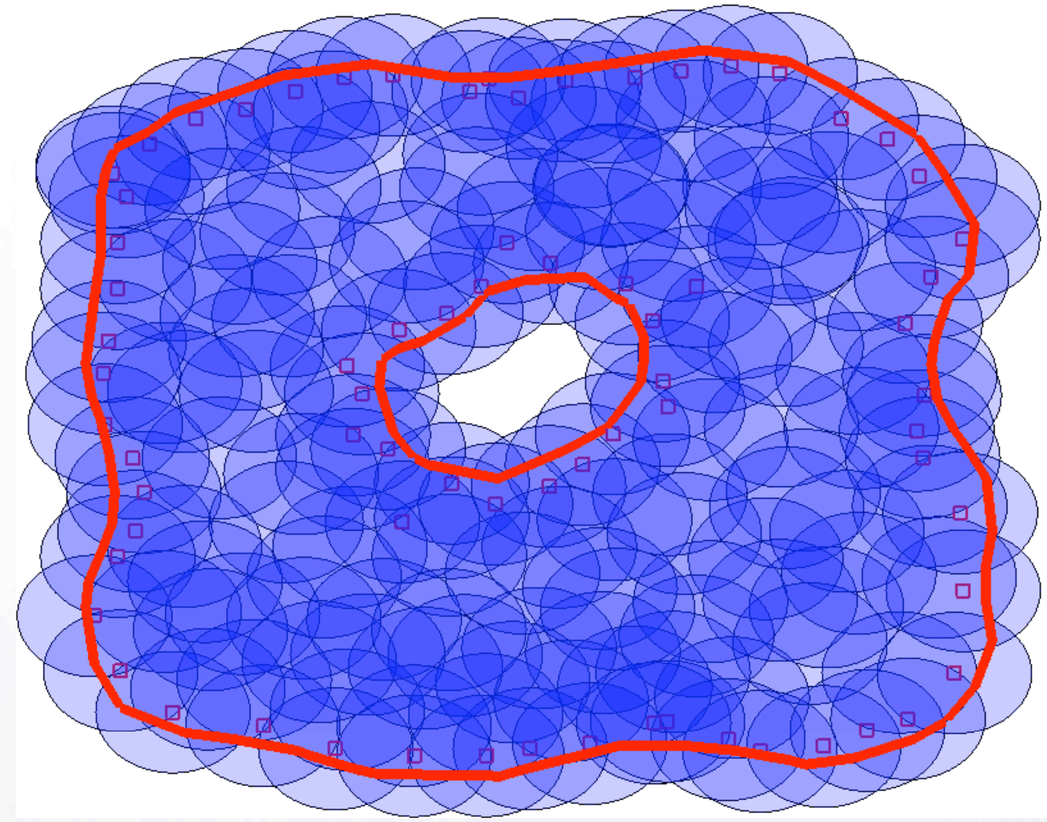




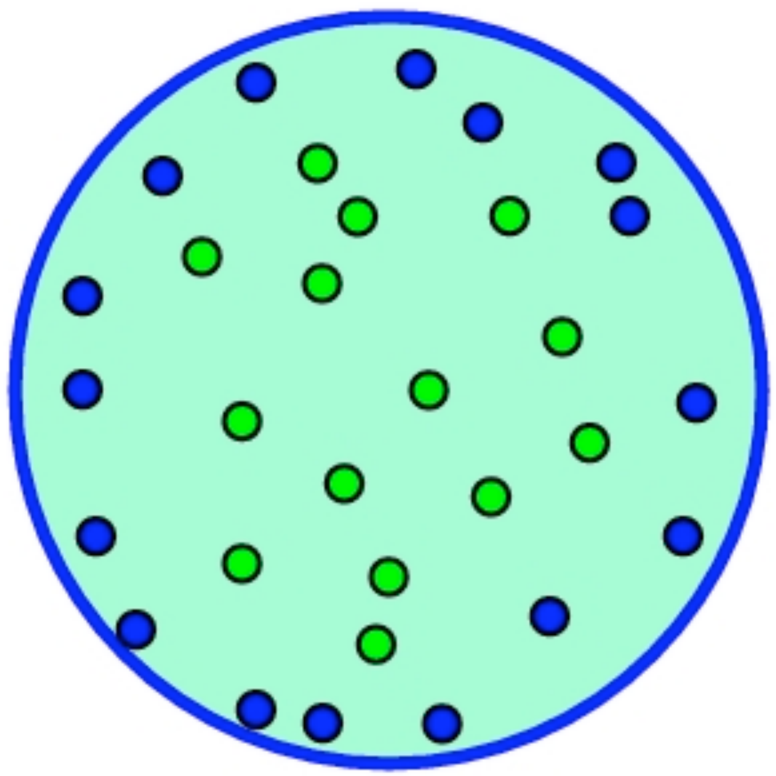
- For a union of disks in the plane we define a "nerve complex"
  - a vertex for every disk
  - an edge whenever two disks intersect
  - a triangle whenever three disks intersect
- ✔ The nerve complex has the same topology as the original union of disks
- ✘ We need precise inter-robot distances to calculate the nerve complex

Example

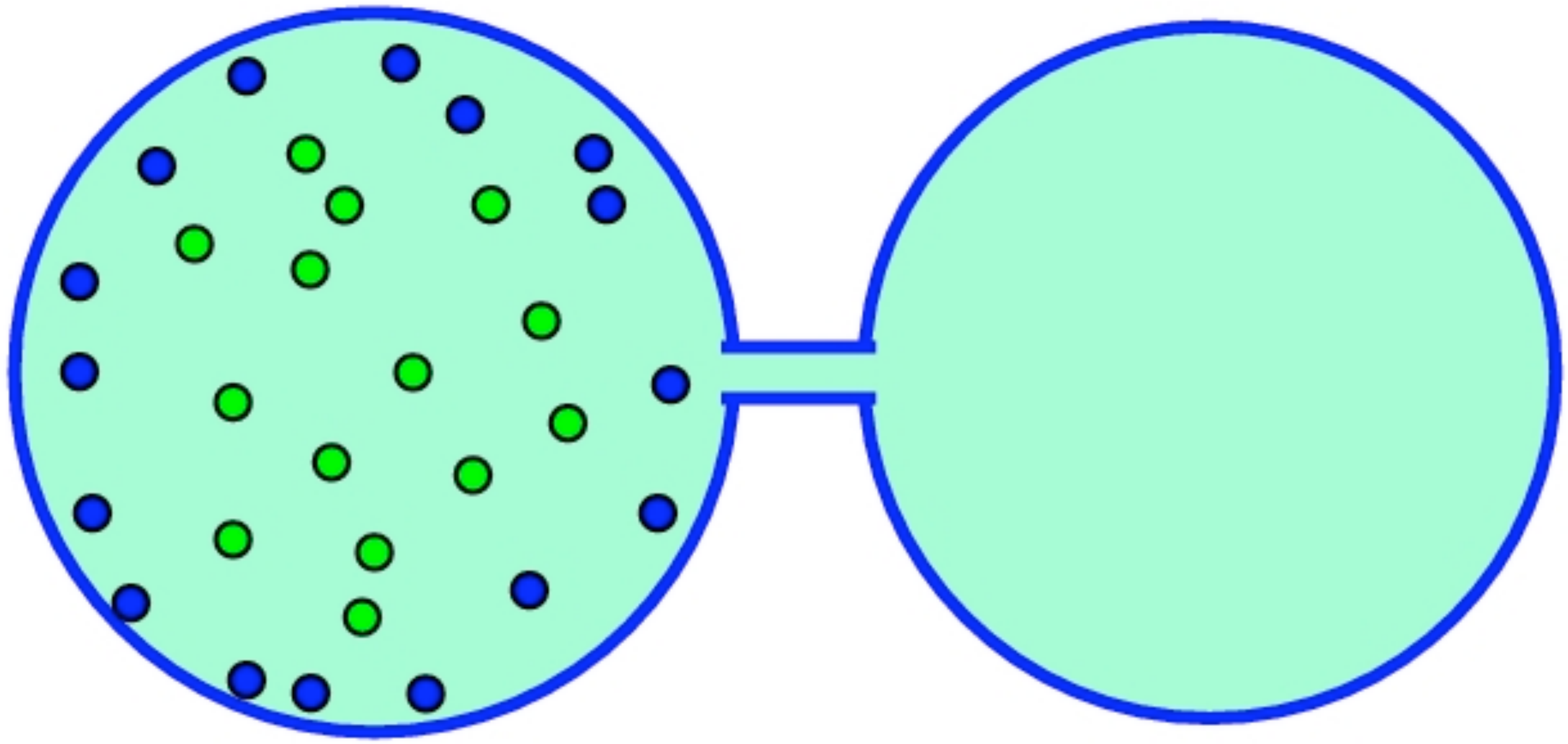




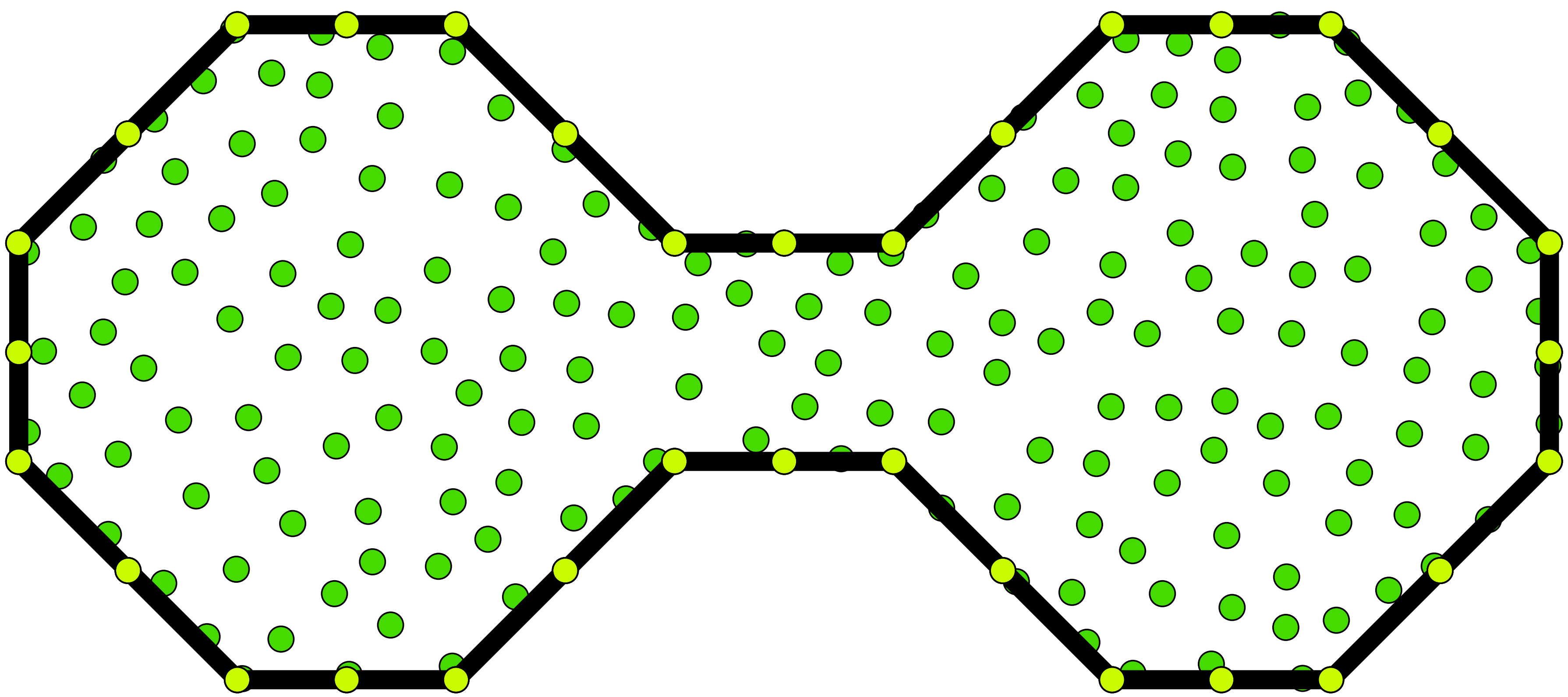
Why we need to avoid pinched domains



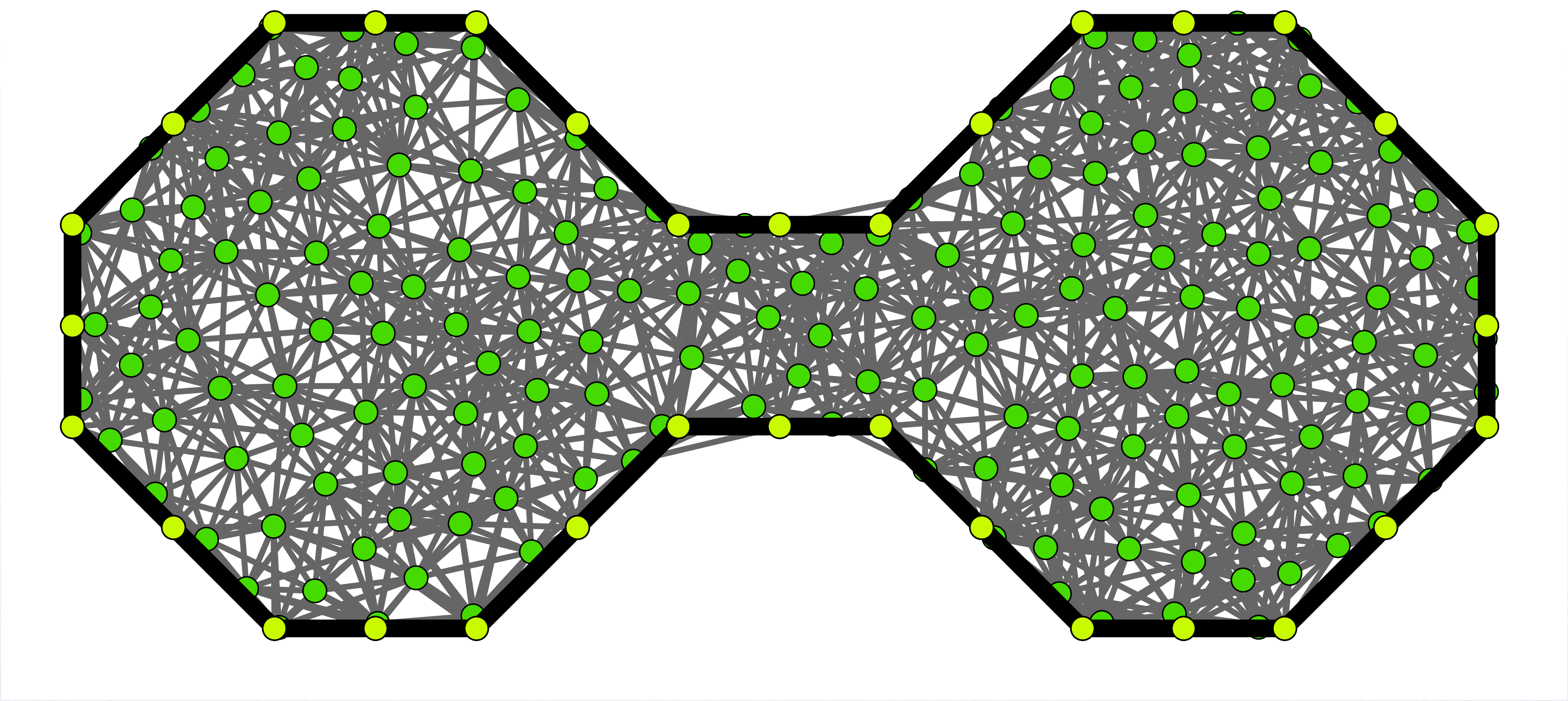
Why we need to avoid pinched domains



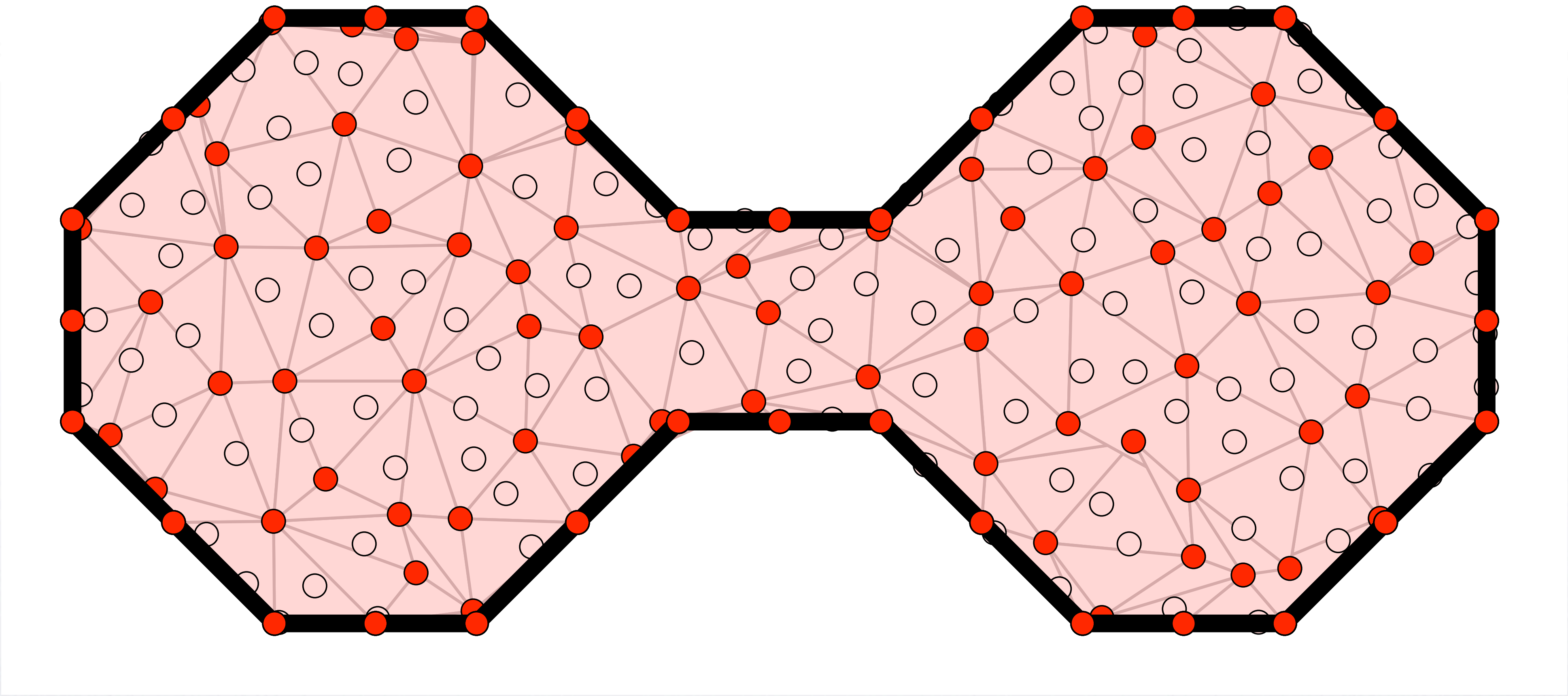
2-dimensional coverage: controlled boundary



2-dimensional coverage: controlled boundary



2-dimensional coverage: controlled boundary



Optimization

- Tahbaz-Salehi & Jadbabaie (2008 results)
  - Finding tight cycles around holes in coverage
    ⇒ identify and repair holes efficiently
  - Finding minimal 2-cycles spanning the region
    ⇒ switch off unused sensor nodes

- Strategy:
  - Replace L^0 optimization with L^1 optimization

$$\underset{\beta \in C_{k+1}}{\operatorname{argmin}} \|\alpha + d\beta\|_{L^0} \qquad \underset{\beta \in C_{k+1}}{\operatorname{argmin}} \|\alpha + d\beta\|_{L^1}$$

  - Solve L^1 problem using subgradient methods
  - Find criteria under which L^1 optimum recovers L^0 optimum

Discrete vs Continuous

Point-cloud topology

- Algebraic topology measures qualitative features of a space X
  - How many components?
  - How many tunnels/voids?
  - How do paths and loops deform within X?
- These are measured by algebraic invariants
  - fundamental group $\pi_1(X)$
  - homology groups $H_k(X)$ and Betti numbers $b_k(X)$
  - products $H_j(X) \times H_k(X) \to H_{j+k}(X)$
- Can we compute these invariants from a finite sample $Y \subset X$?

Standard Pipeline (first attempt)

hidden/unknown space X

↓

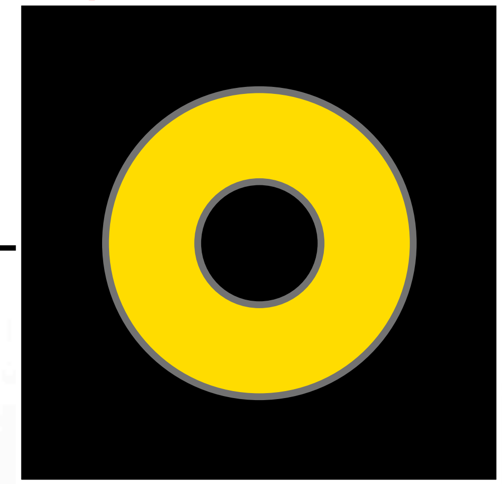
finite sample $Y \subset X$

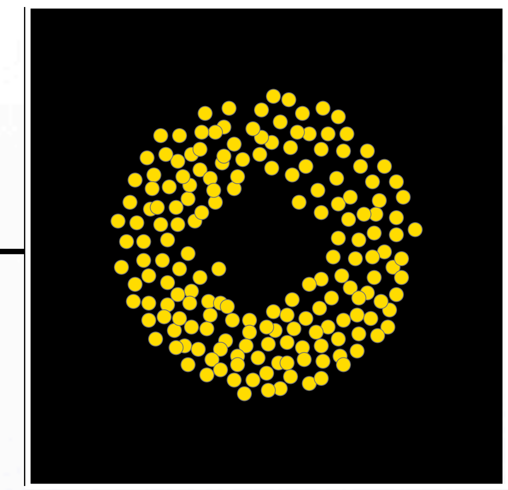
↓

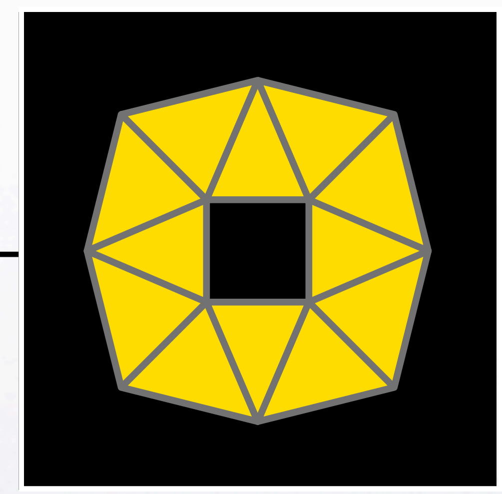
simplicial complex $S = S(Y)$

↓

homology invariants of S







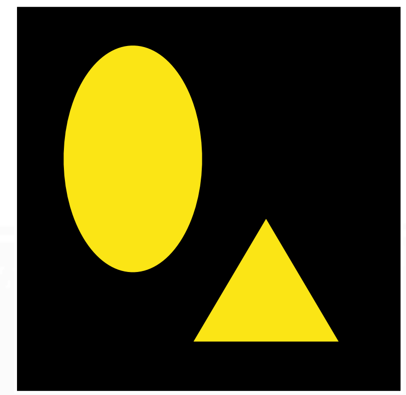
$b_0 = 1$

$b_1 = 1$

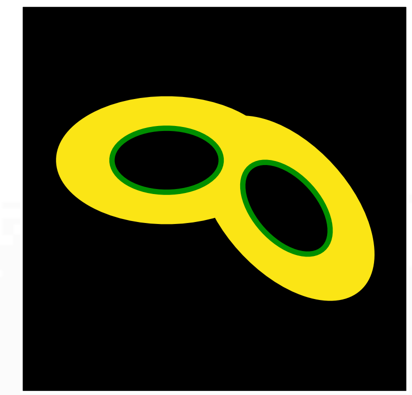
$b_2 = 0$

Betti numbers ↔ features

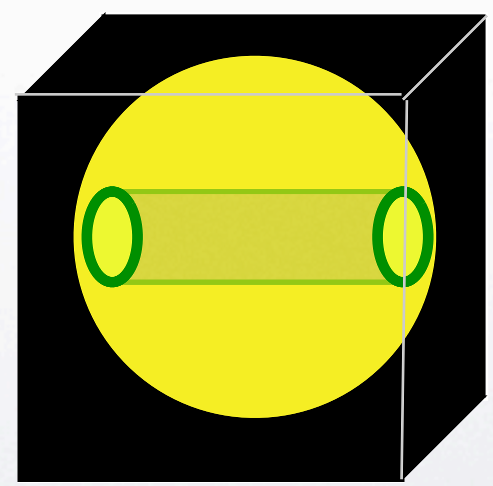
- For an object in 2D space
  - b_0 is the number of components
  - b_1 is the number of holes
- For an object in 3D space
  - b_0 is the number of components
  - b_1 is the number of tunnels or handles
  - b_2 is the number of voids
- (and so on, in higher dimensions)



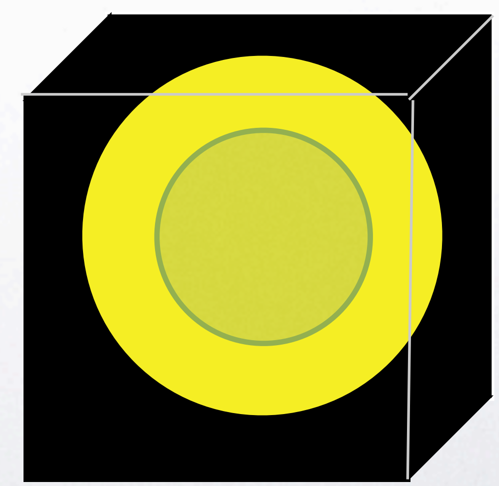
$b_0 = 2$, $b_1 = 0$



$b_0 = 1$, $b_1 = 2$



$b_0 = 1$, $b_1 = 1$, $b_2 = 0$



$b_0 = 1$, $b_1 = 0$, $b_2 = 1$

Reconstruction theorems

- Various constructions for S(Y)
  - Cech complex (folklore)
  - Rips–Vietoris complex (folklore)
  - α-shape complex (Edelsbrunner, Mücke)
  - strong/weak witness complexes (Carlsson, dS)
- Desire theorems of the form:

If Y is well-sampled from X then S(Y) ≈ X

- e.g. Niyogi–Smale–Weinberger (2004), Cech complex

Discrete vs continuous

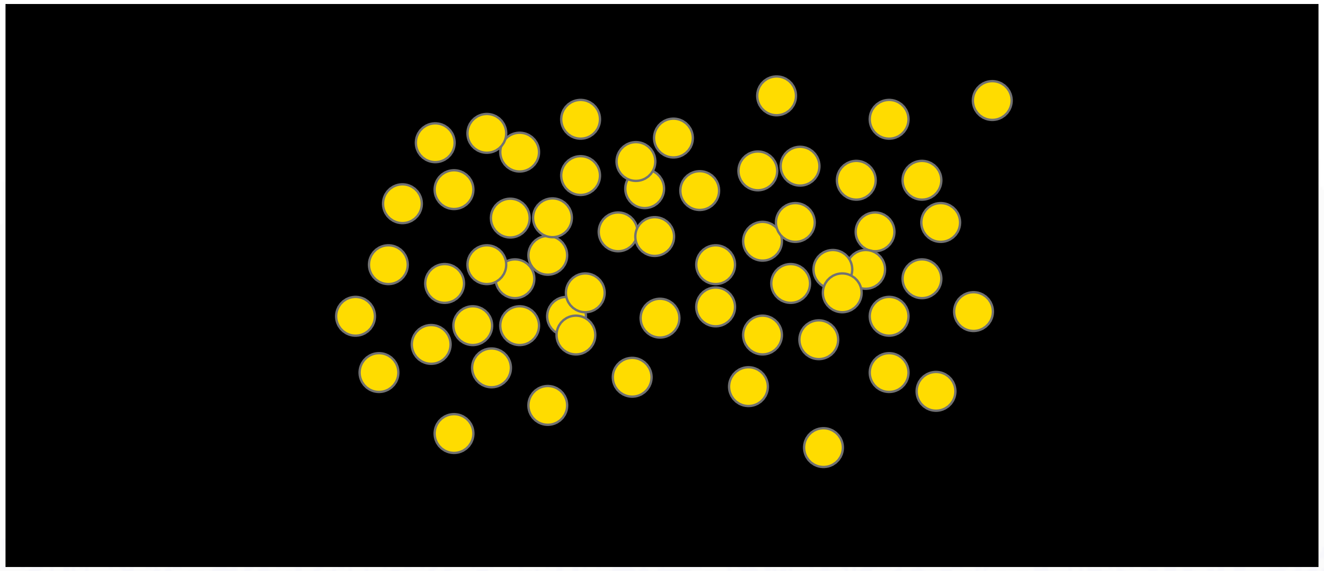
- Betti numbers are discrete

- Topological spaces
  - topological spaces are continuous
  - the space of topological spaces is discrete

- Finite point-clouds
  - point-clouds are discrete
  - the space of point-clouds is continuous

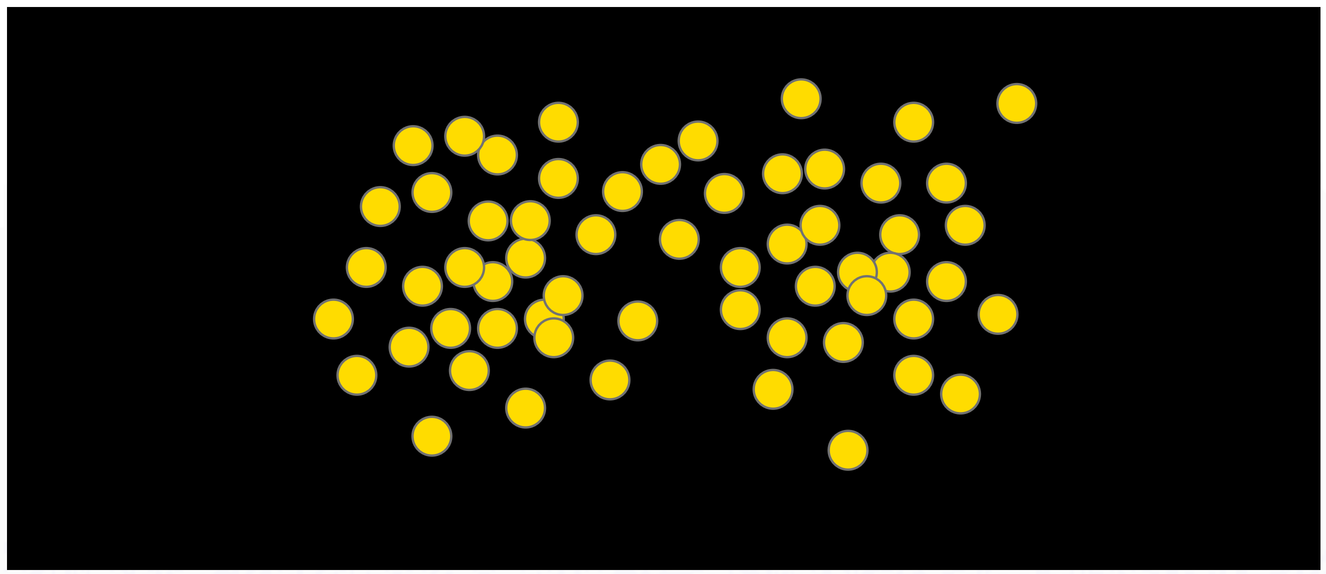
- Therefore, raw Betti numbers are
  - ✔ very handy for topological spaces
  - ✘ a bit dangerous for point-clouds

One lump or two?



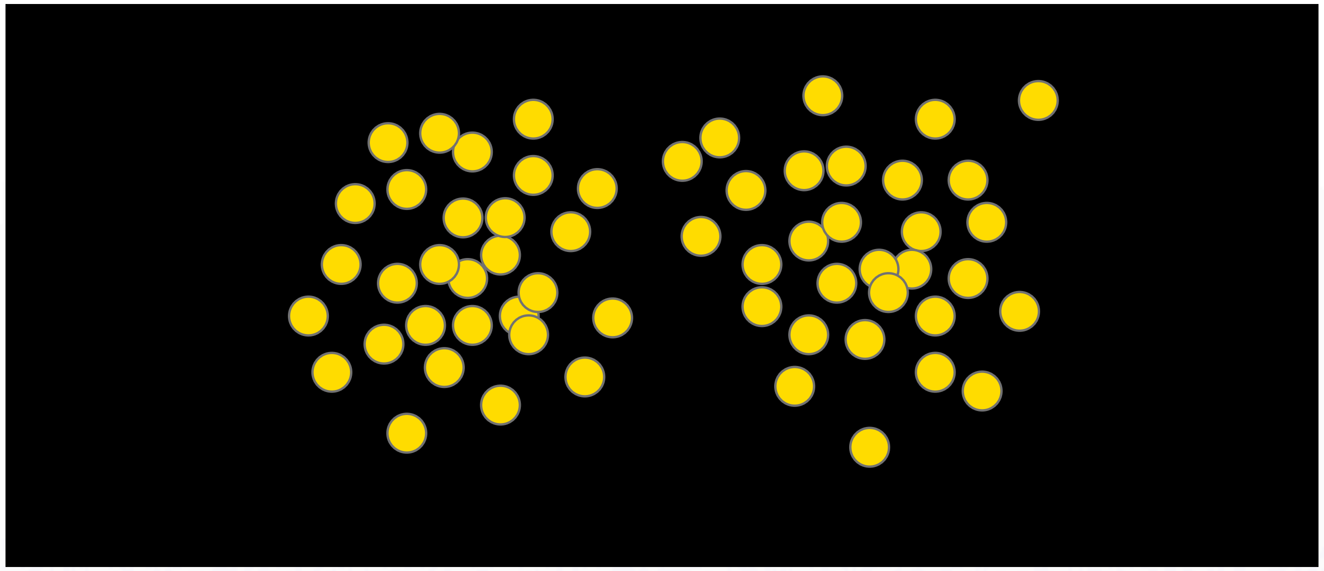
At which parameter value does the number of components change?

One lump or two?



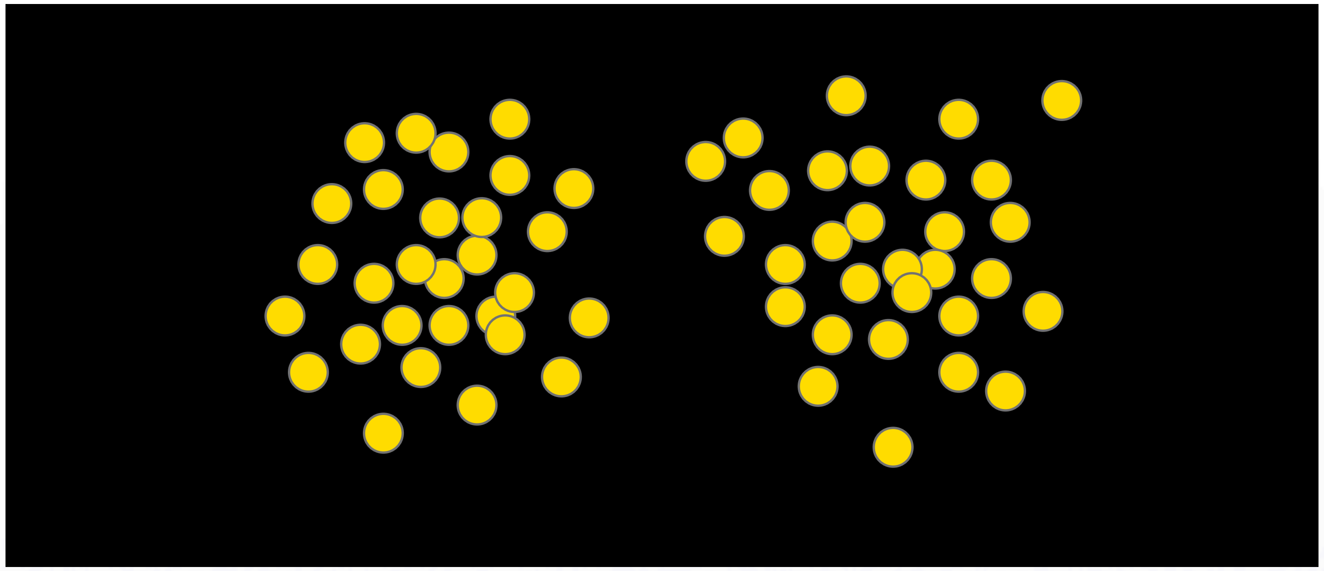
At which parameter value does the number of components change?

One lump or two?



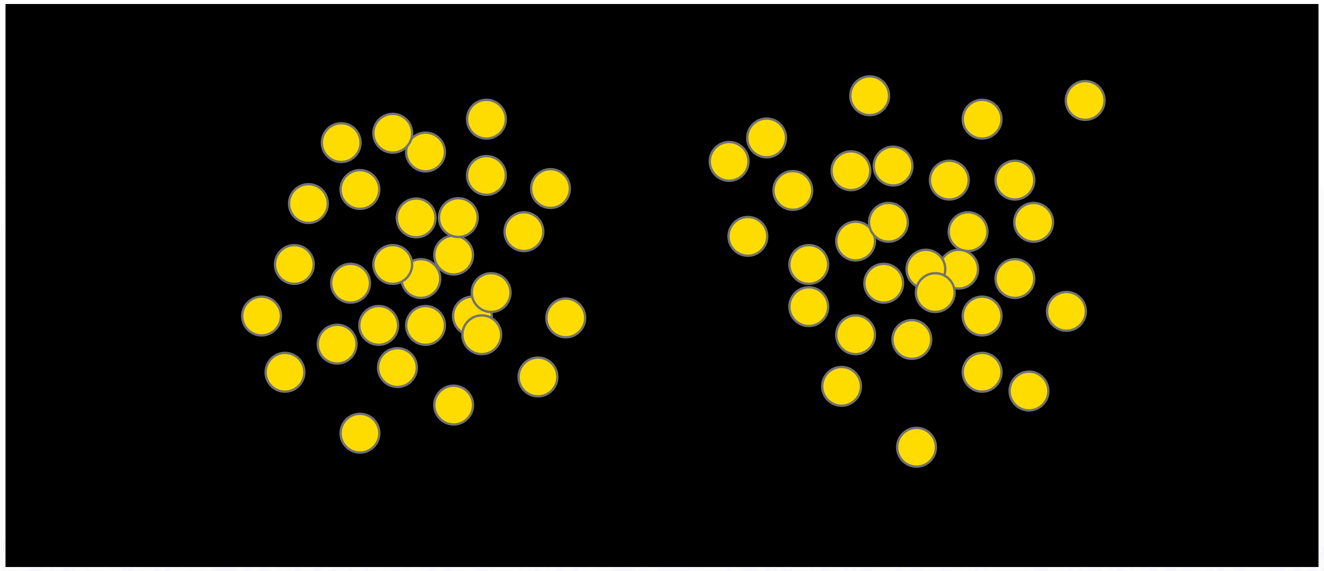
At which parameter value does the number of components change?

One lump or two?



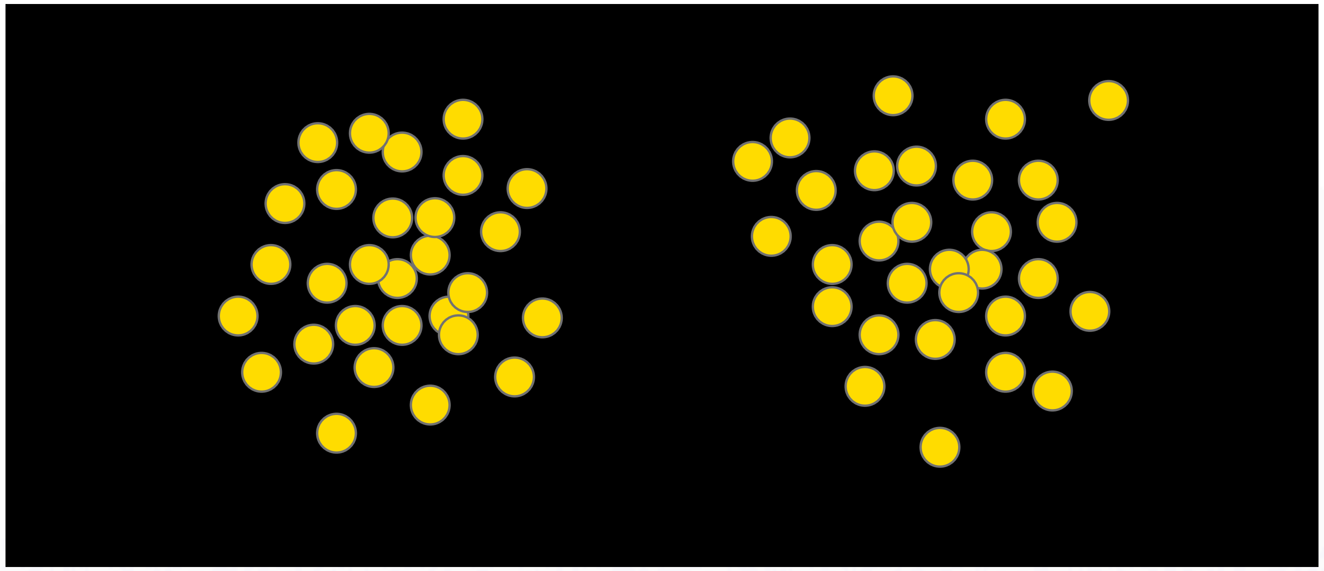
At which parameter value does the number of components change?

One lump or two?



At which parameter value does the number of components change?

One lump or two?



At which parameter value does the number of components change?

Standard Pipeline (second attempt)

hidden/unknown space X

↓

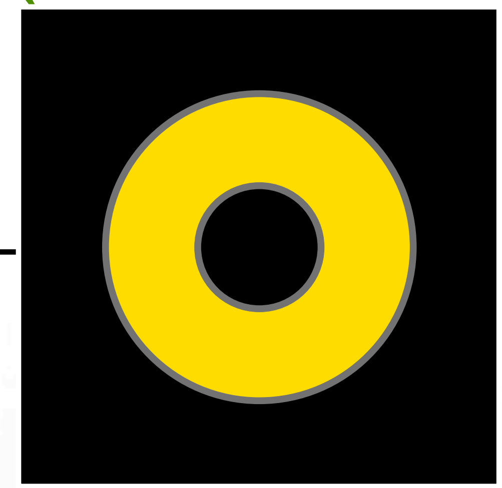
finite sample Y⊂X

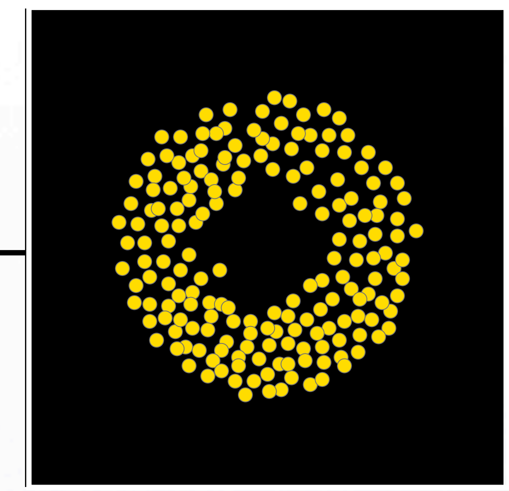
↓

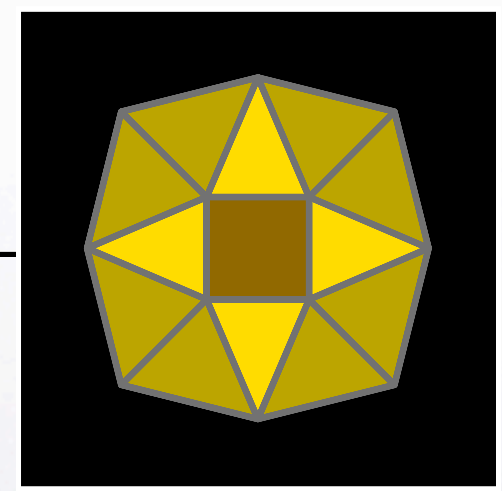
labelled complex $S(r) = S(Y,r)$

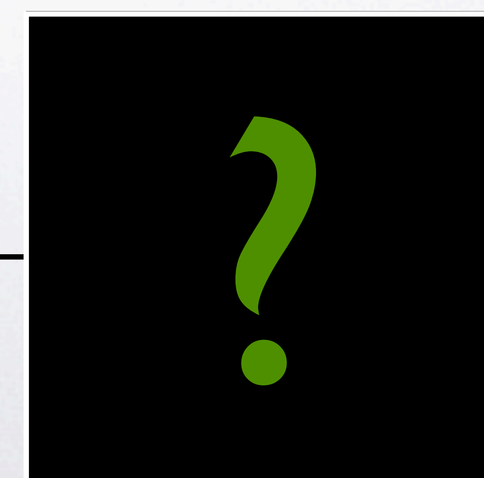
↓

quantitative topology









Persistence

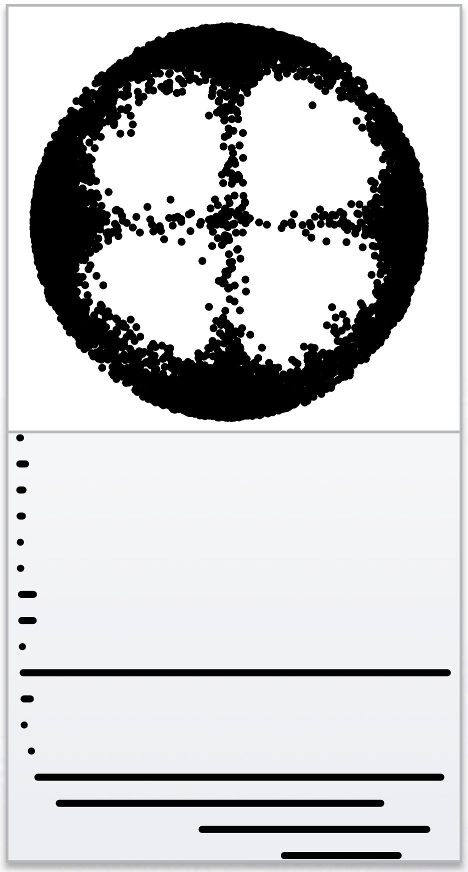
- Monotone increasing family of spaces

$$\mathbf{X} = \{X_\epsilon \mid \epsilon \geq 0\} \quad \text{such that} \quad X_\epsilon \subseteq X_{\epsilon'} \text{ if } \epsilon \leq \epsilon'$$

- Persistent homology

$$\text{rank}\,[H_*(X_\epsilon) \to H_*(X_{\epsilon'})] \quad \text{for all } \epsilon \leq \epsilon'$$

- Barcode (Edelsbrunner, Letscher, Zomorodian '00)
  - finite collection of intervals $[b_i,d_i)$
  - [b,d) indicates feature born at time b, dies at time d



- Stability theorem (Cohen-Steiner, Edelsbrunner, Harer '07)
  - barcode depends continuously on the underlying data
  - see also Chazal, Cohen-Steiner, Glisse, Guibas, Oudot '09

- Continuous measurements (interval length) coupled to discrete information (number of intervals

Persistence pipeline

hidden/unknown space X

↓

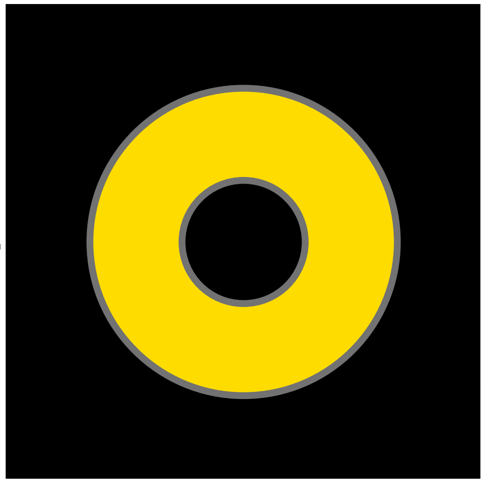
finite sample Y⊂X

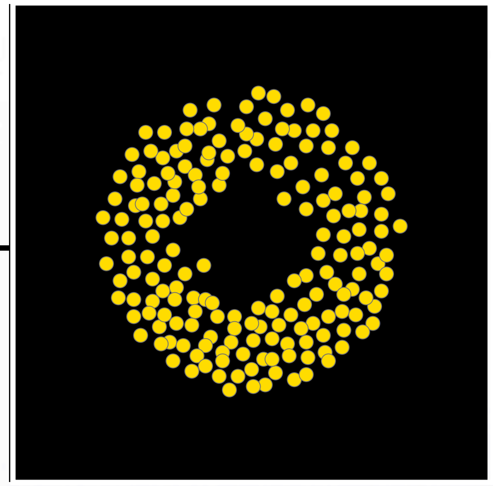
↓

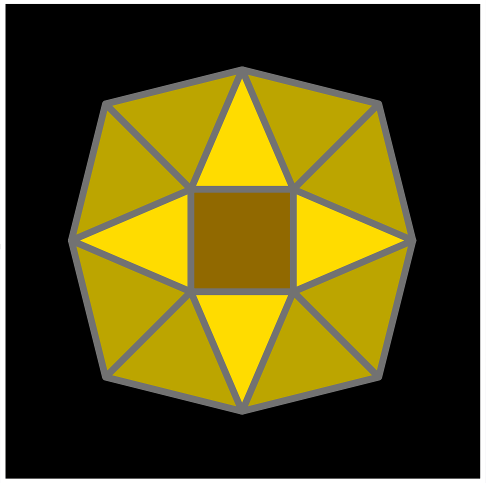
filtered complex S(r) = S(Y,r)

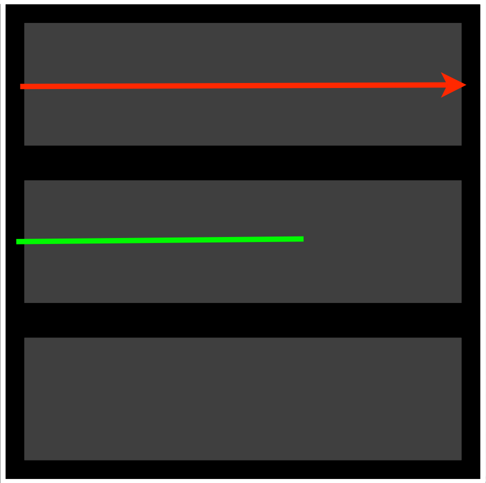
↓

persistent homology of S(r)





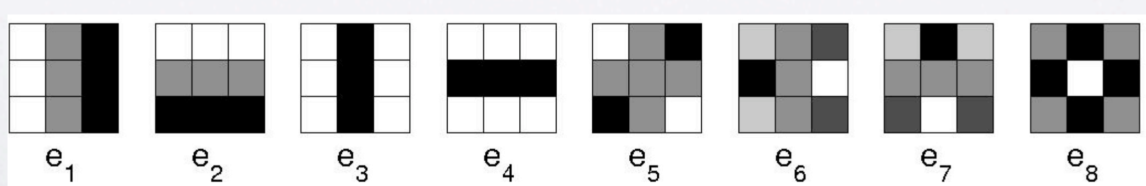




Visual Image Patches

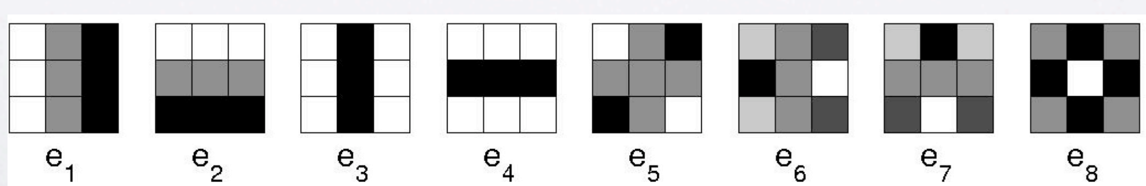
Visual image patches

- Lee, Pedersen, Mumford (2003) studied the local statistical properties of natural images (from Van Hateren's database)
- 3-by-3 pixel patches with high contrast between pixels: are some patches more likely than others?
- Carlsson, VdS, Ishkhanov, Zomorodian (2004/8): topological properties of high-density regions in pixel-patch space



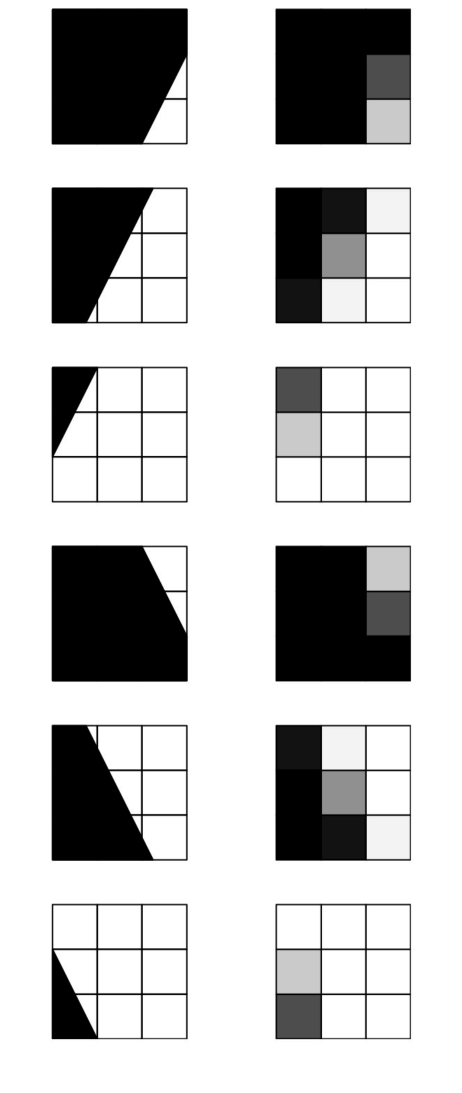
The space of image patches

- ~4.2 million high-contrast 3-by-3 patches selected randomly from images in database.
- Normalise each patch twice: subtract mean intensity, then rescale to unit norm.
- Normalised patches live on a unit 7-sphere in 8-dimensional space with the following basis:



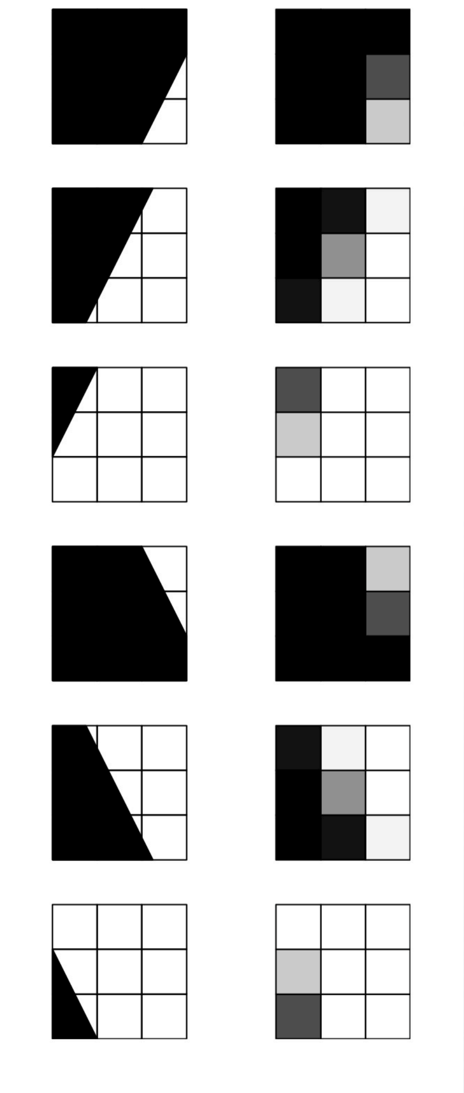
High-density regions

- LPM2003 found that the distribution of patches is dense in the 7-sphere.
- There are high-density regions:
  - edge features
- Can we describe the structure of the high-density regions?
  - threshold by k-nearest-neighbour density estimator



High-density regions

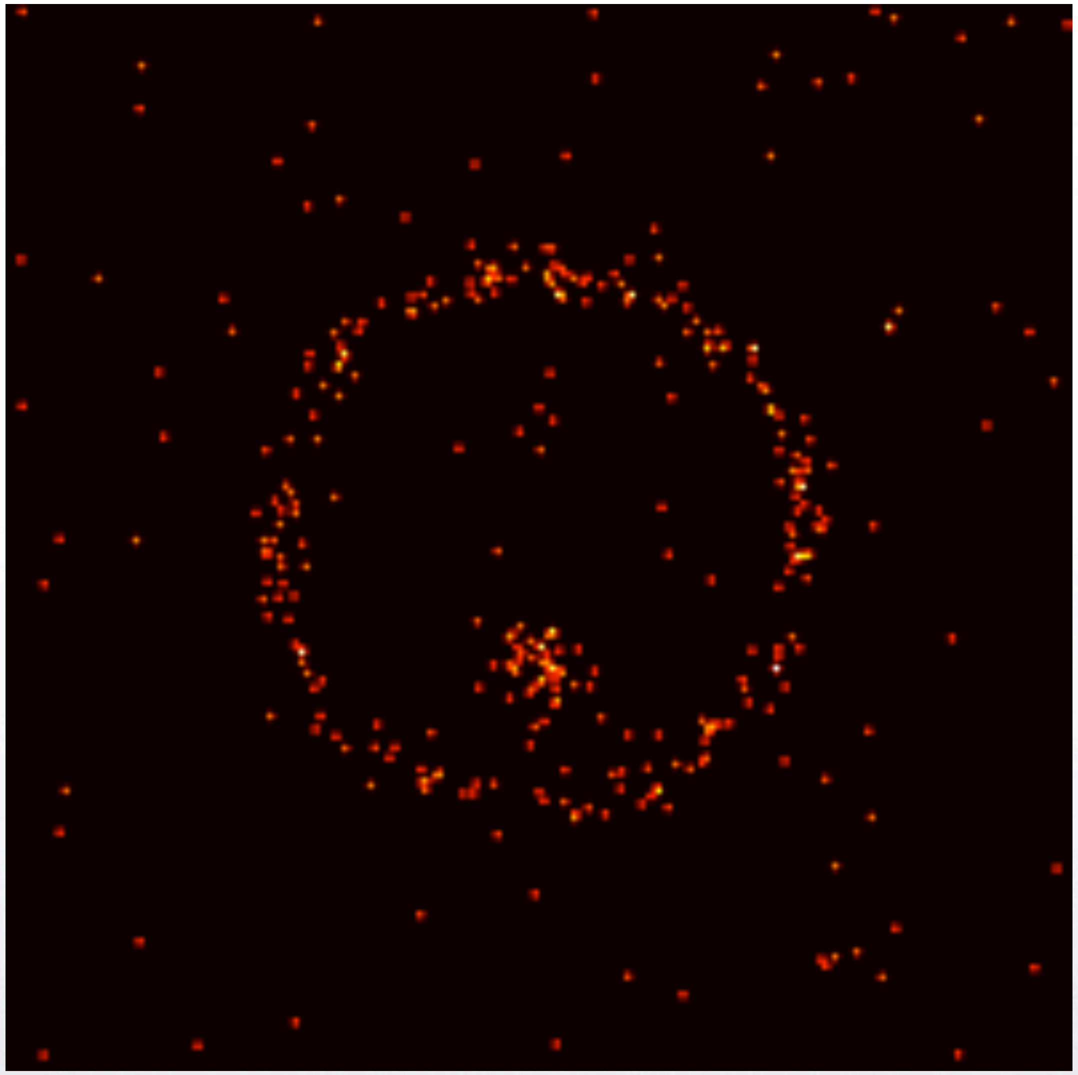
- LPM2003 found that the distribution of patches is dense in the 7-sphere.
- There are high-density regions:
  - edge features
- Can we describe the structure of the high-density regions?
  - threshold by k-nearest-neighbour density estimator



Straining a data soup

Varying the density parameter

(toy example)

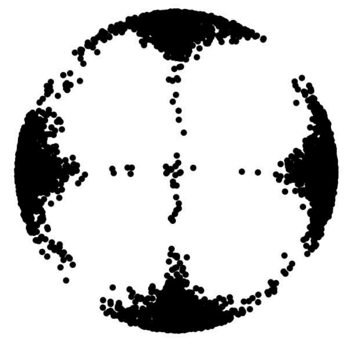
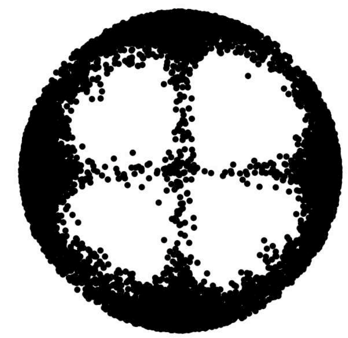
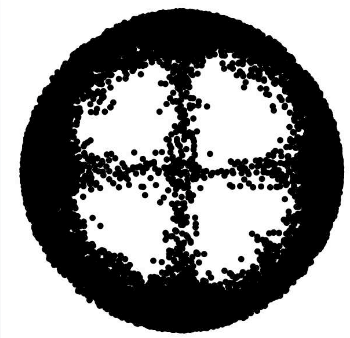
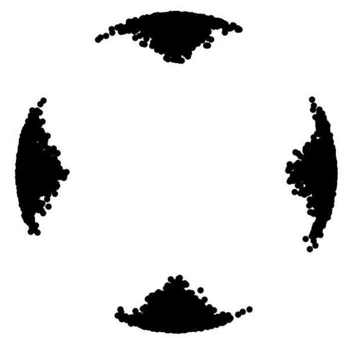
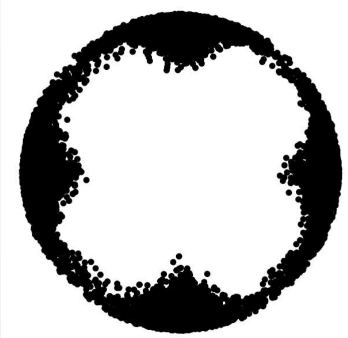
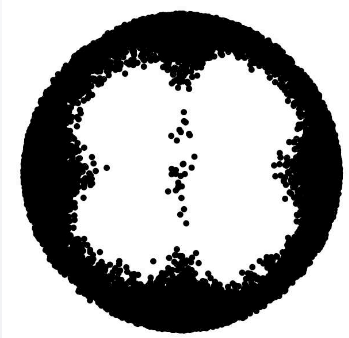
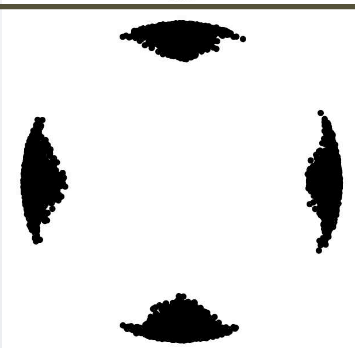
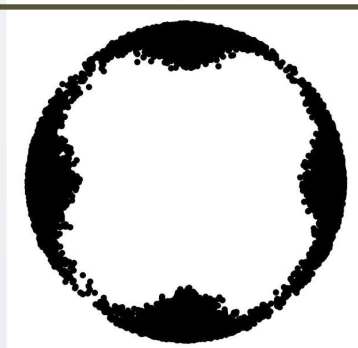
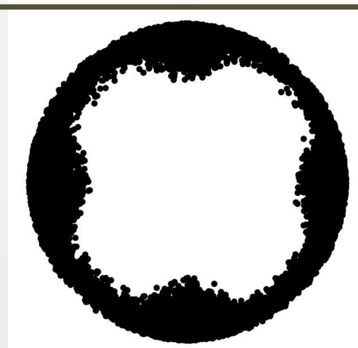


Varying the density parameter

(toy example)

Straining a data soup

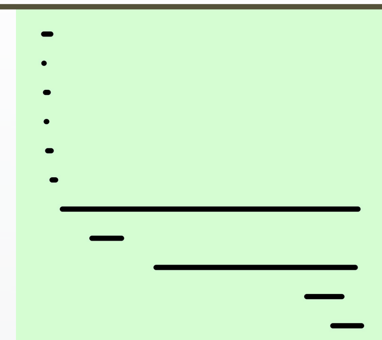
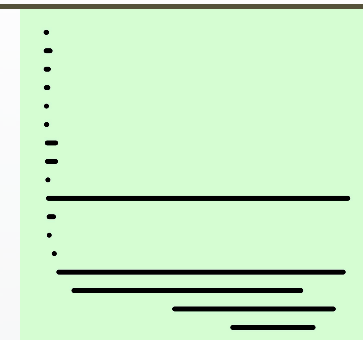
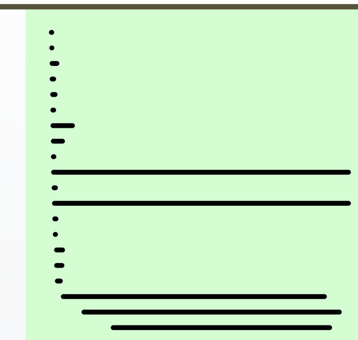
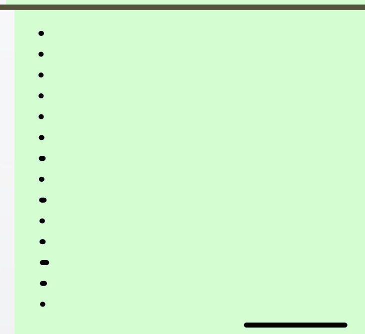
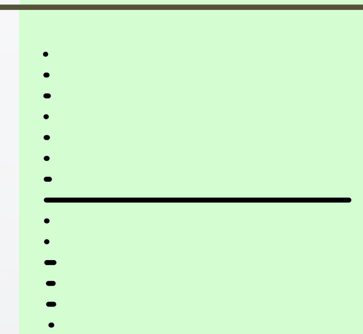
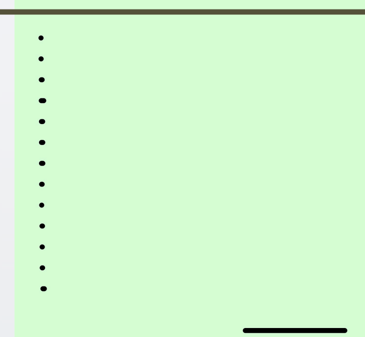
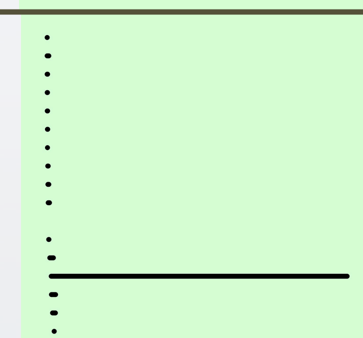
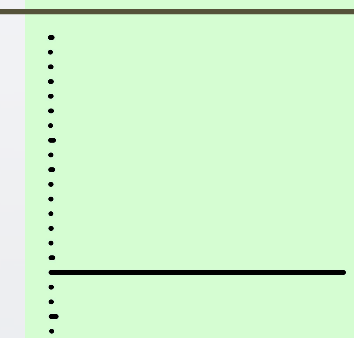
A small platter of cuts

	10%	20%	30%
K=15			
K=100			
K=300			

Vin de Silva
http://pages.pomona.edu/~vds04747/

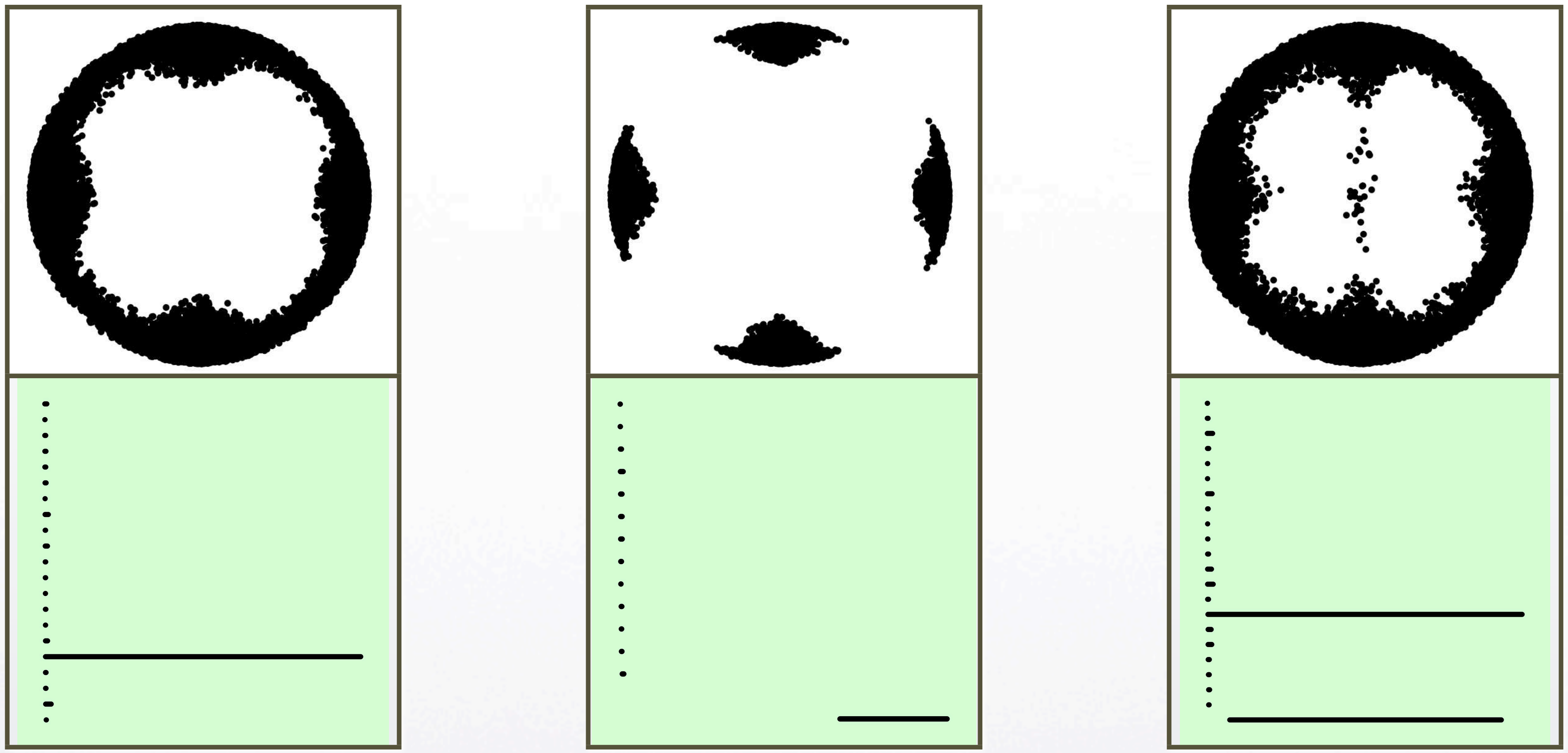
Topology in the 21st Century
January 26-27, 2009

A small platter of cuts

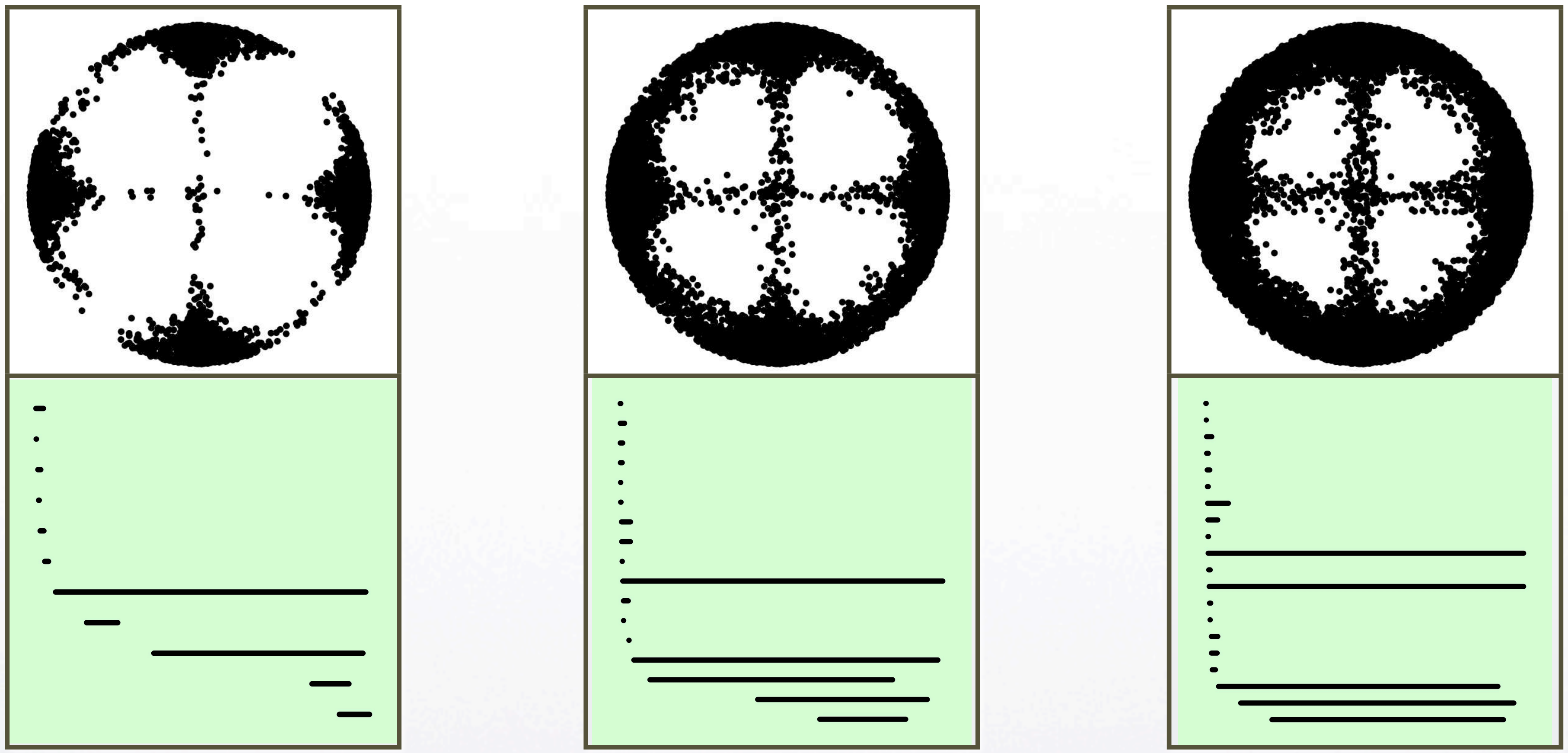
	10%	20%	30%
K=15			
K=100			
K=300			

Vin de Silva
http://pages.pomona.edu/~vds04747/

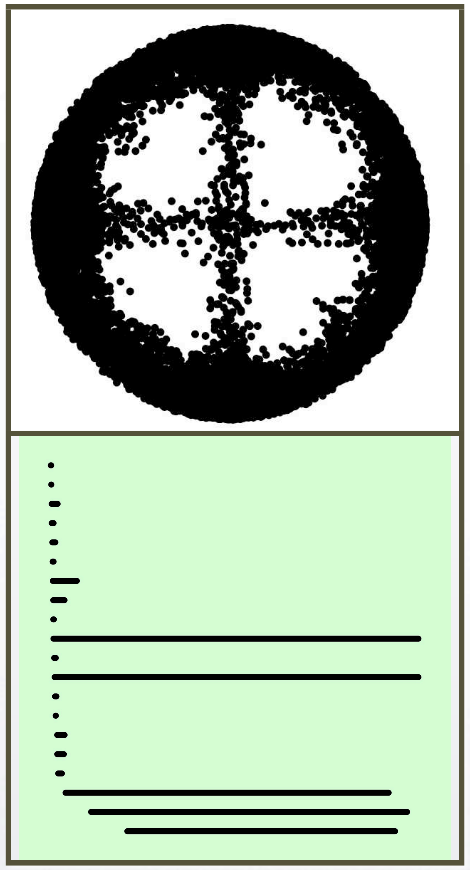
(8-dimensional data)

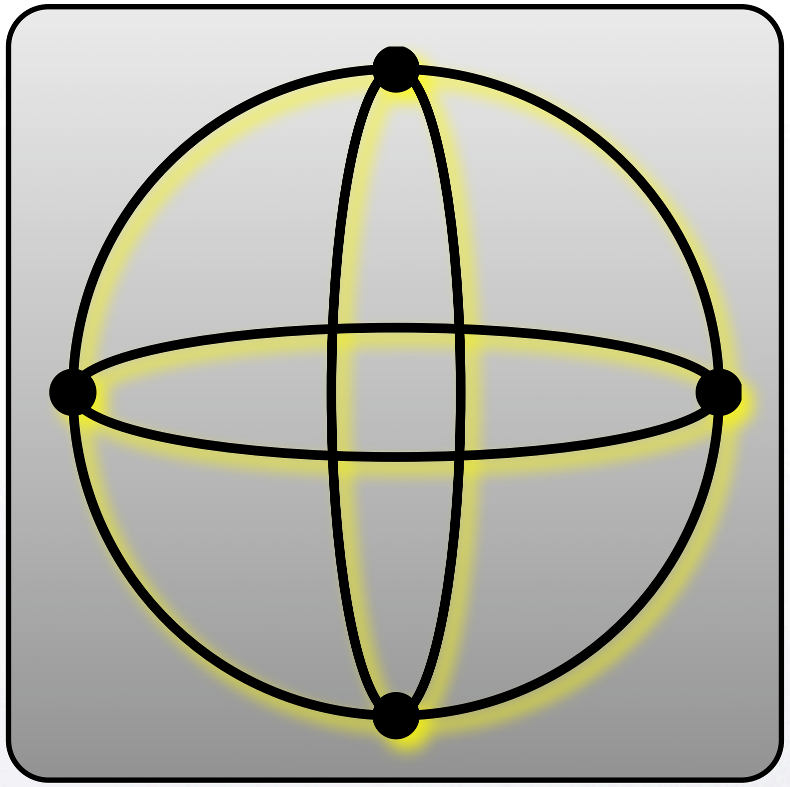


(8-dimensional data)

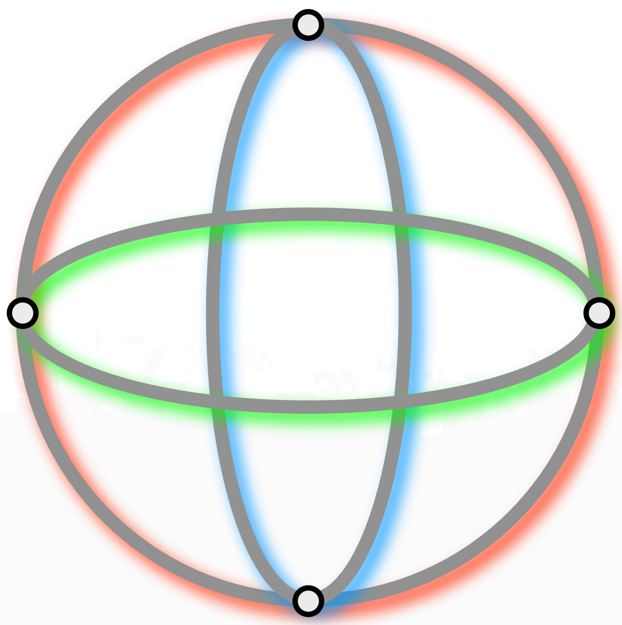


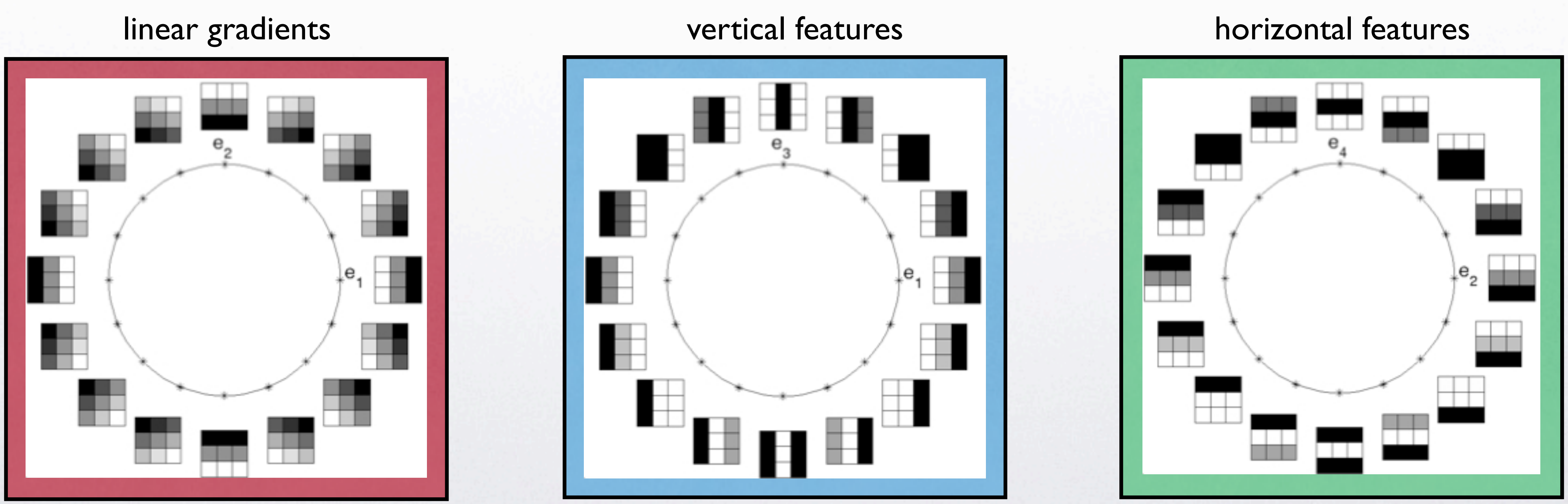
3-circles model





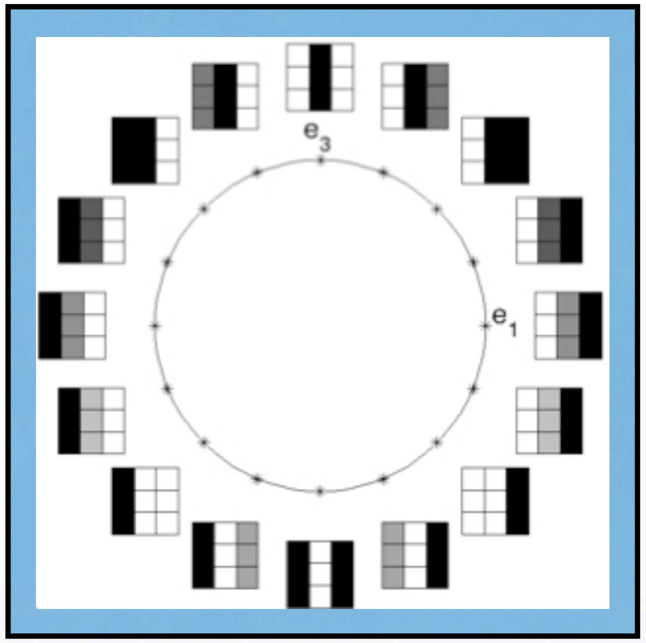
3 circles explained



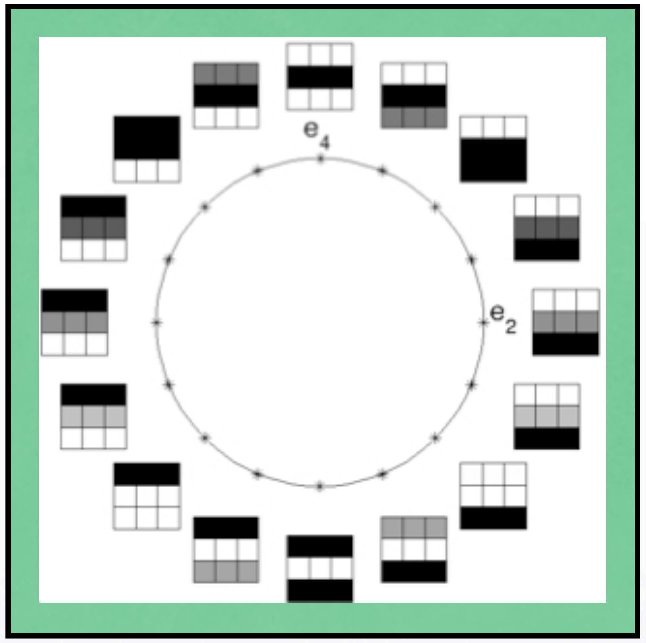


The secondary circles

vertical features



horizontal features

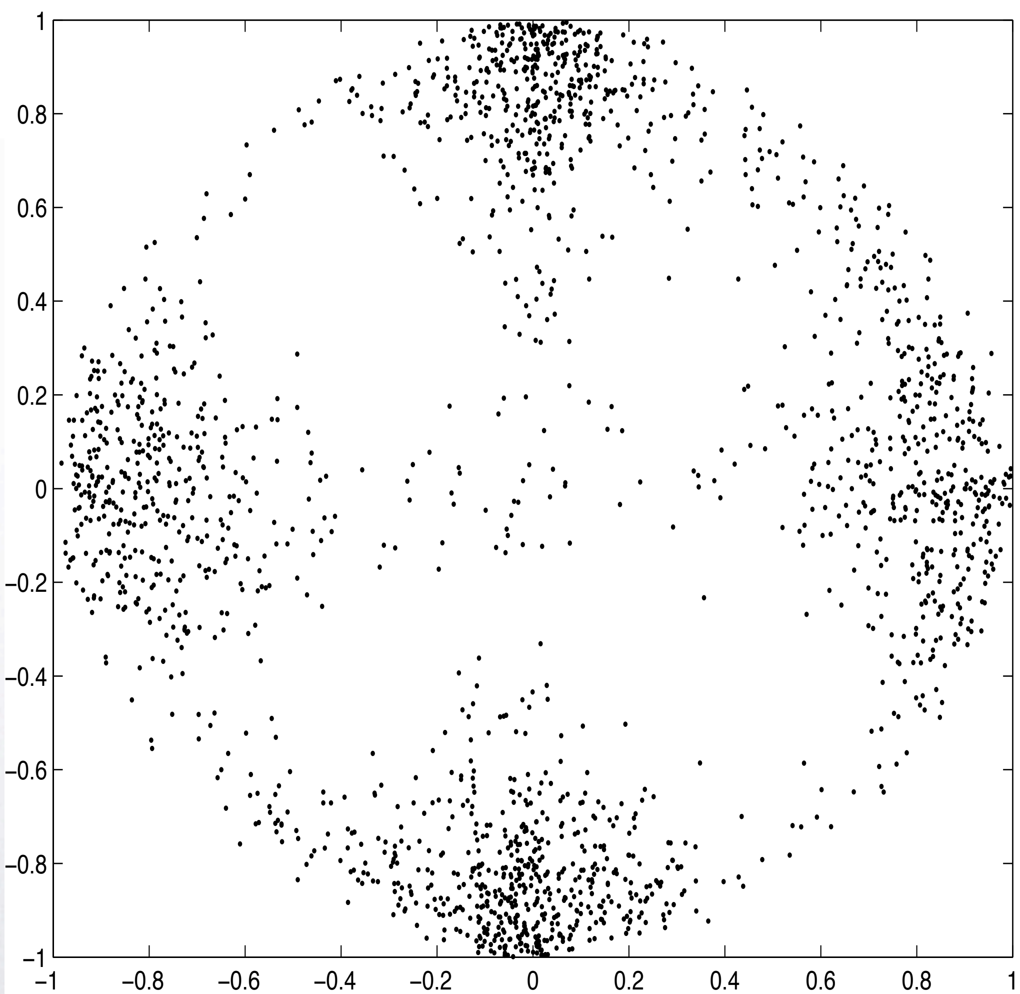


Why is there a predominance of vertical/horizontal local features?

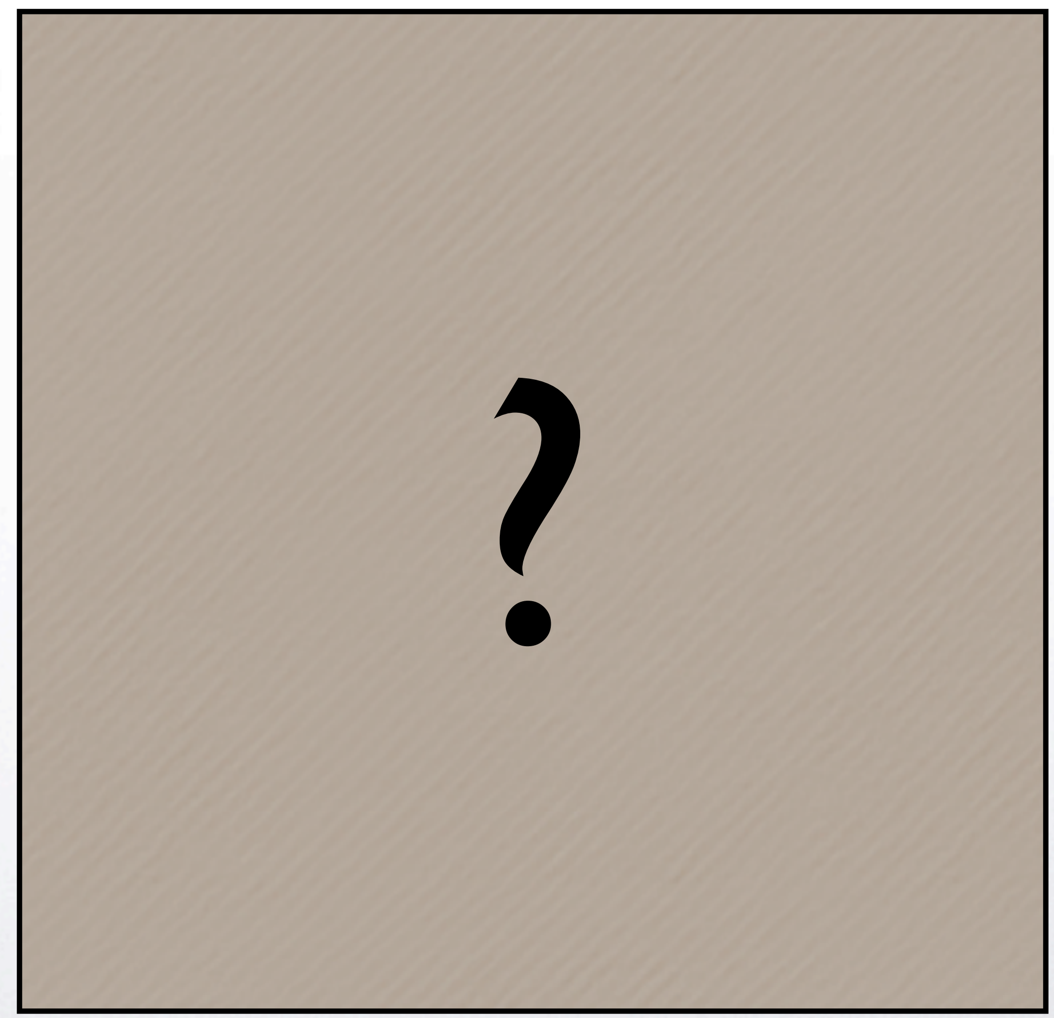
Artefact of the square patch shape?
Artefact of the natural world?

Tilting the camera

orthogonal images



diagonal images



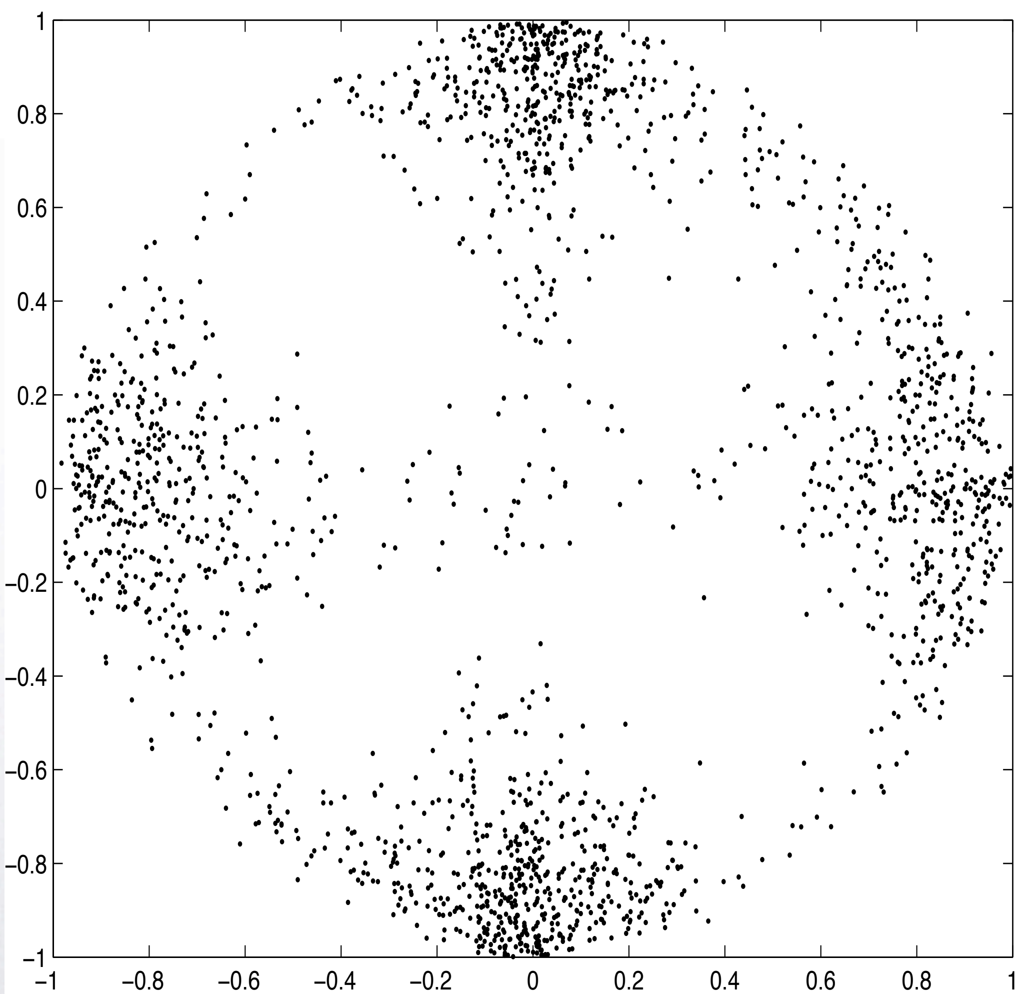
Tilting the camera

orthogonal images

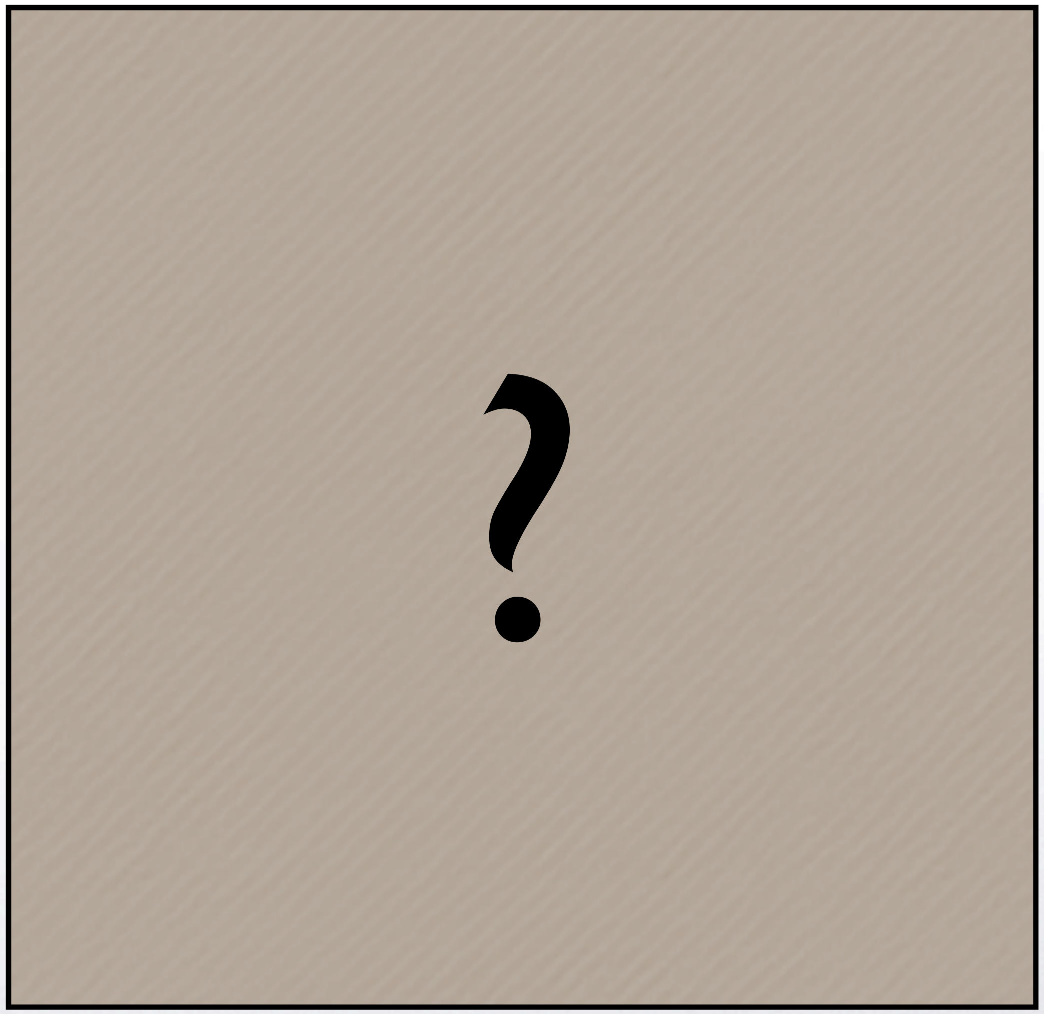
diagonal images

Tilting the camera

orthogonal images

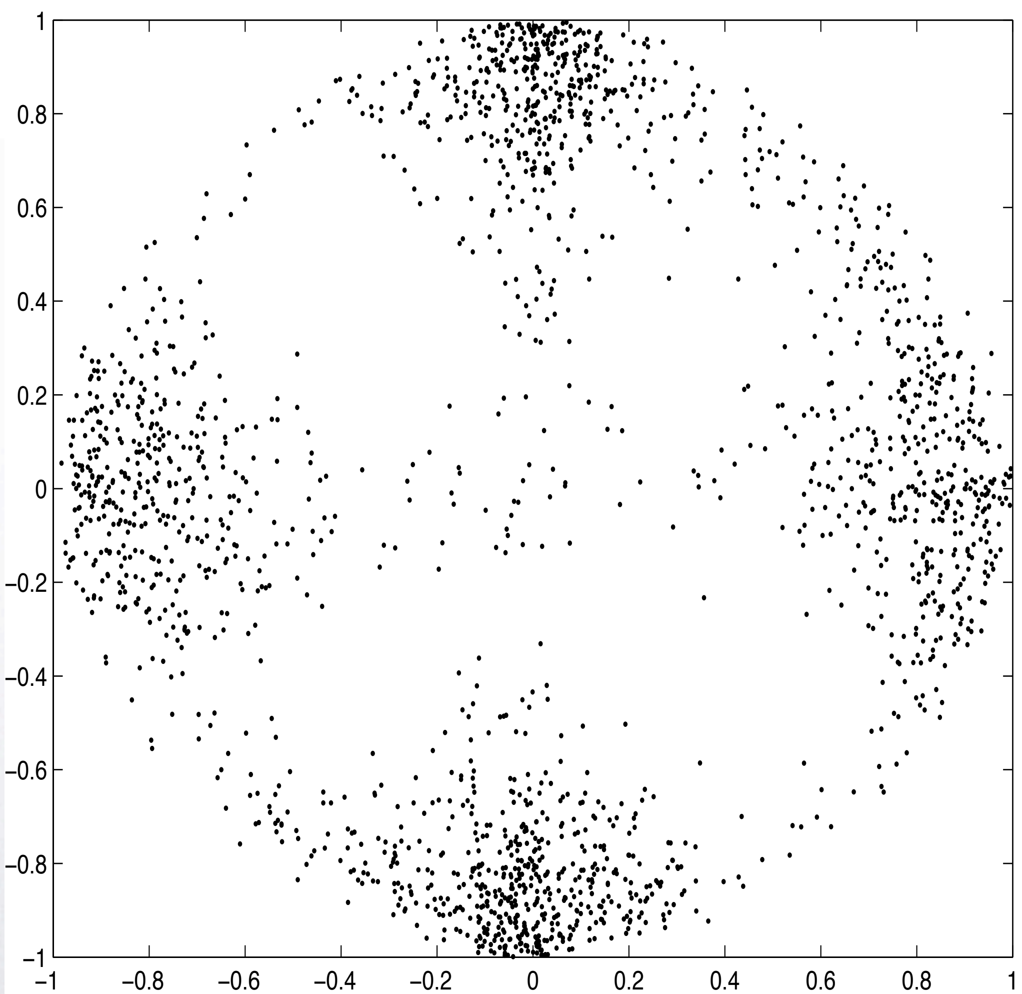


diagonal images

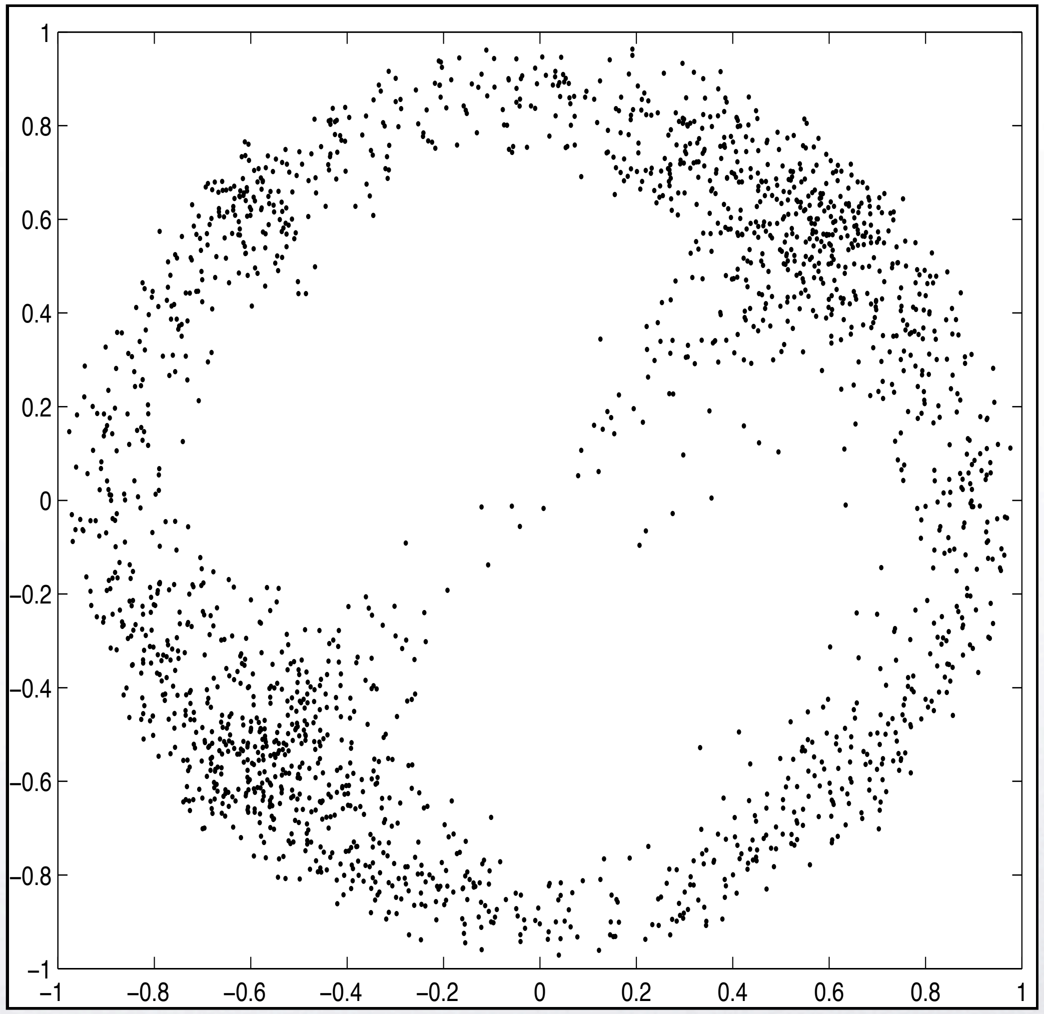


Tilting the camera

orthogonal images



diagonal images



Homogenizing over all tilt angles

- e_1-e_2 circle: arbitrary linear functions ax+by in the image plane.
- e_1-e_3 circle: quadratic functions of x.
- e_3-e_4 circle: quadratic functions of y.
- What about quadratic functions of arbitrary linear functions ax+by?

Witness complexes

Persistence pipeline

hidden/unknown space X

↓

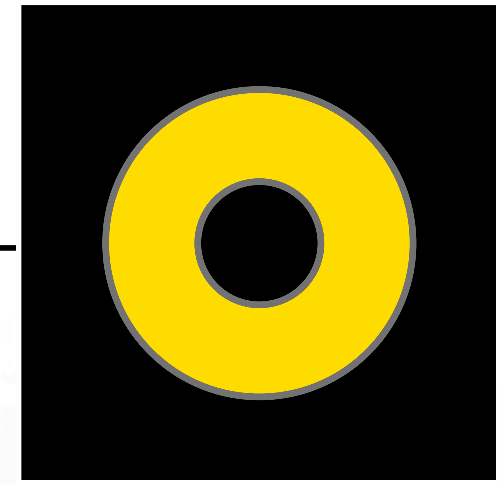
finite sample Y⊂X

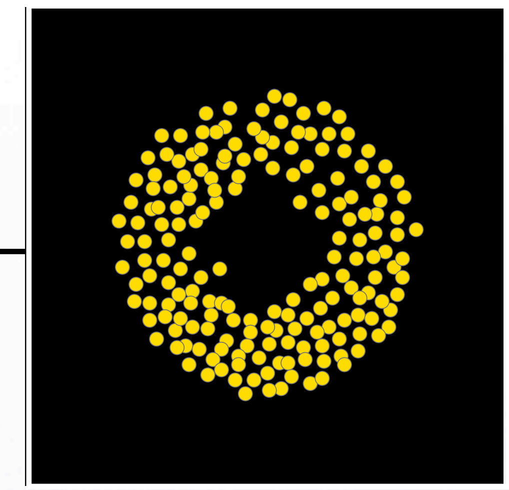
↓

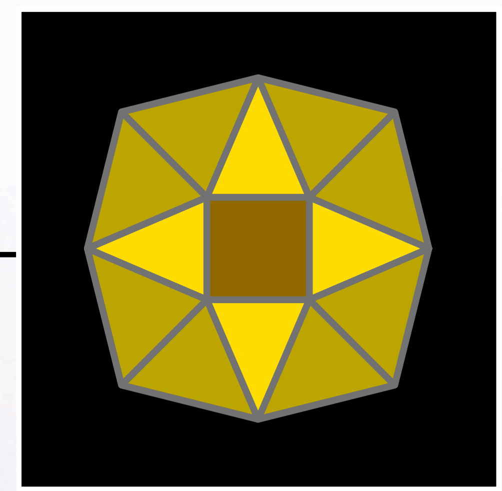
filtered complex S(r) = S(Y,r)

↓

persistent homology of S(r)

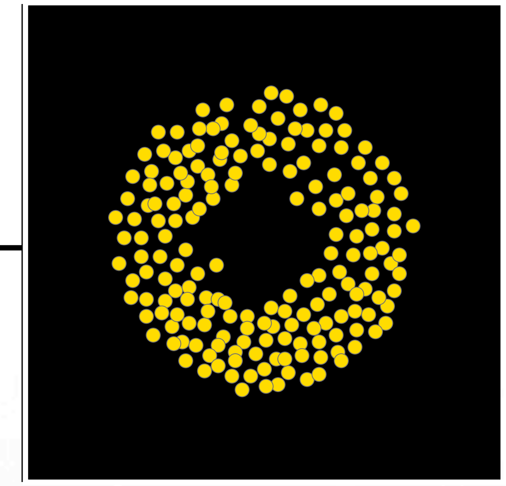




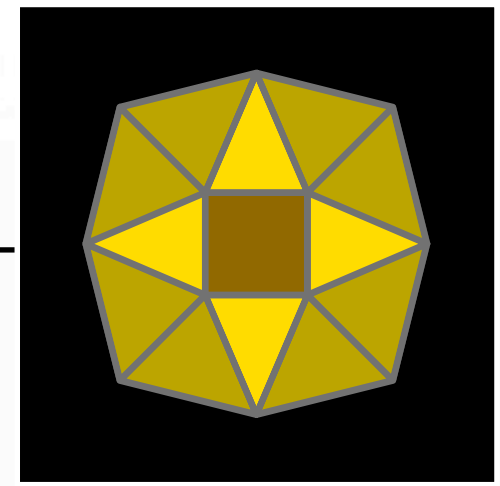


Persistence pipeline

finite sample
Y⊂X



filtered complex
S(r) = S(Y,r)



- Cech complex

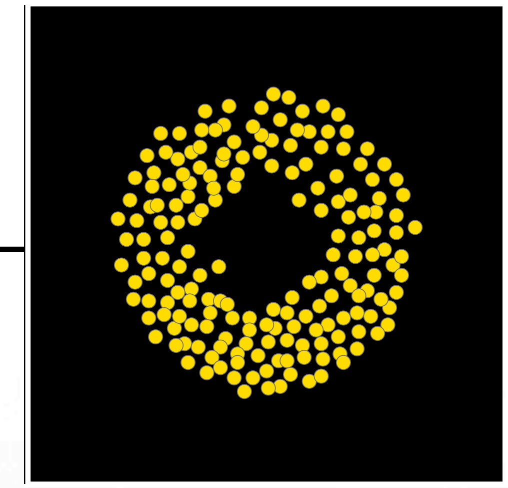
$$\sigma = [a_0, \ldots, a_k] \in \check{\mathrm{C}}\mathrm{ech}(X, \epsilon) \Leftrightarrow \bigcap_{i=0}^{k} B_\epsilon(a_i) \neq \emptyset$$

- Rips complex

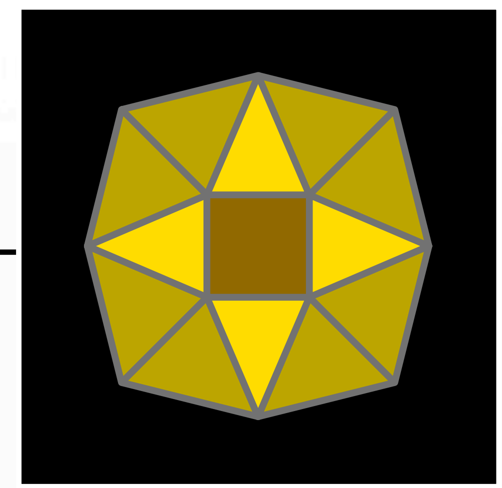
$$\sigma = [a_0, \ldots, a_k] \in \mathrm{Rips}(X, \epsilon) \Leftrightarrow |a_i - a_j| \leq \epsilon, \forall i, j$$

Persistence pipeline

finite sample
$Y \subset X$



filtered complex
$S(r) = S(Y, r)$



- Cech complex

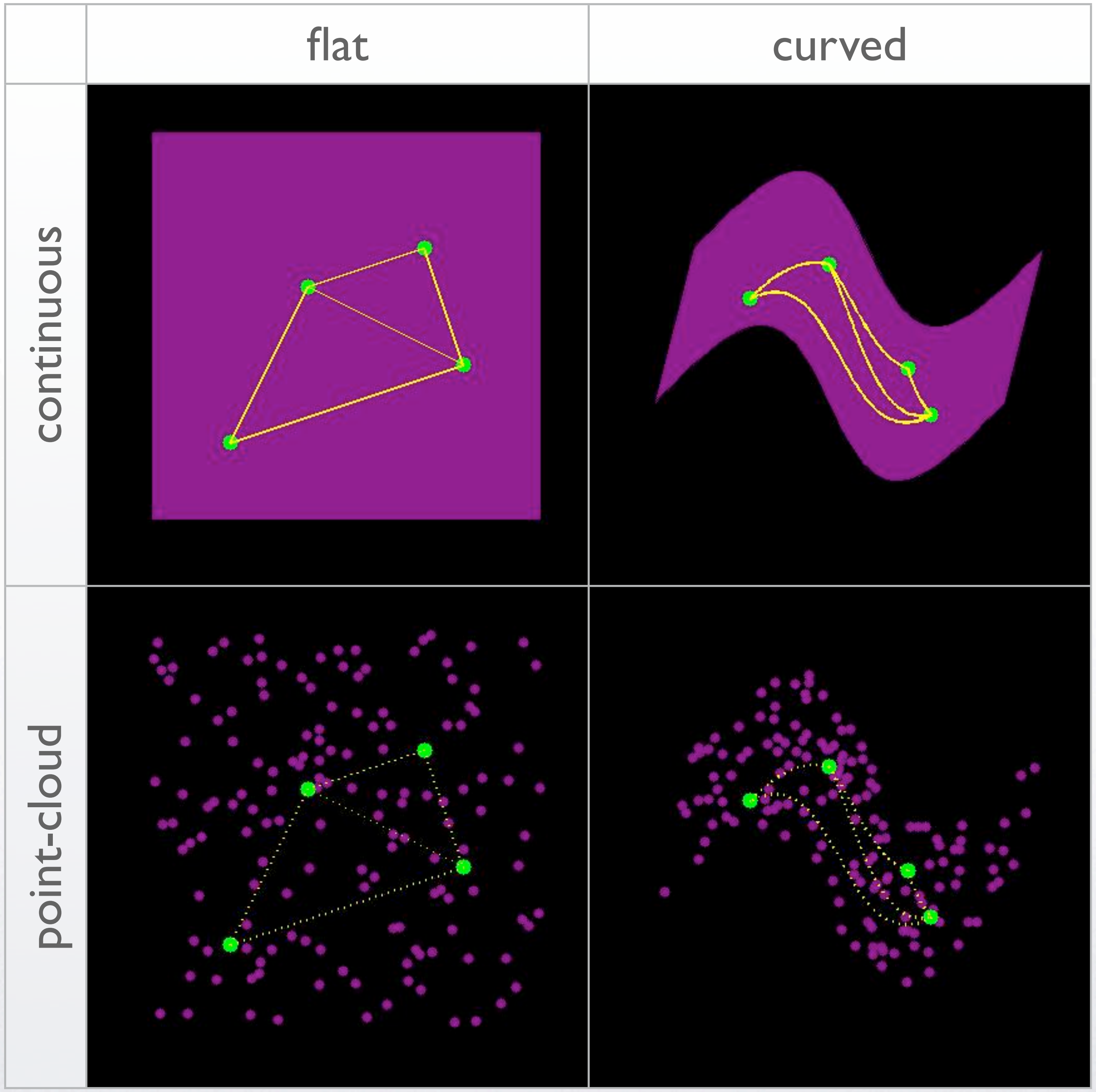
$$\sigma = [a_0, \ldots, a_k] \in \check{\mathrm{C}}\mathrm{ech}(X, \epsilon) \Leftrightarrow \bigcap_{i=0}^{k} B_\epsilon(a_i) \neq \emptyset$$

- Rips complex

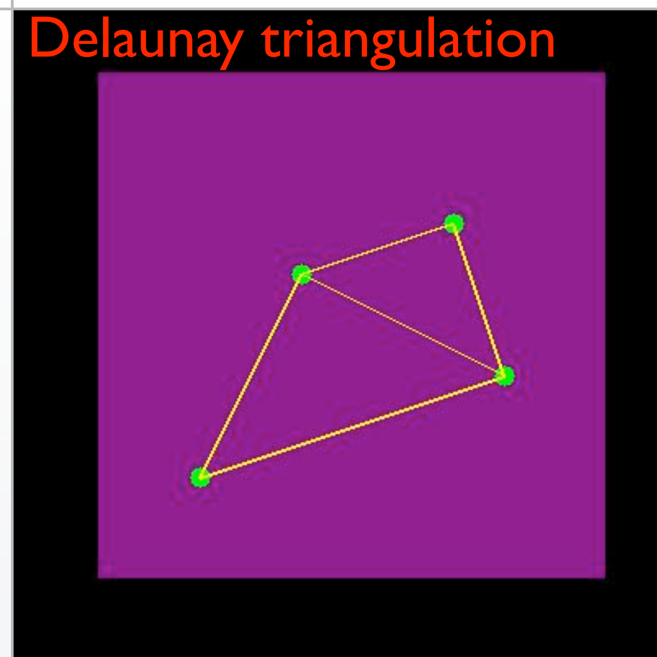
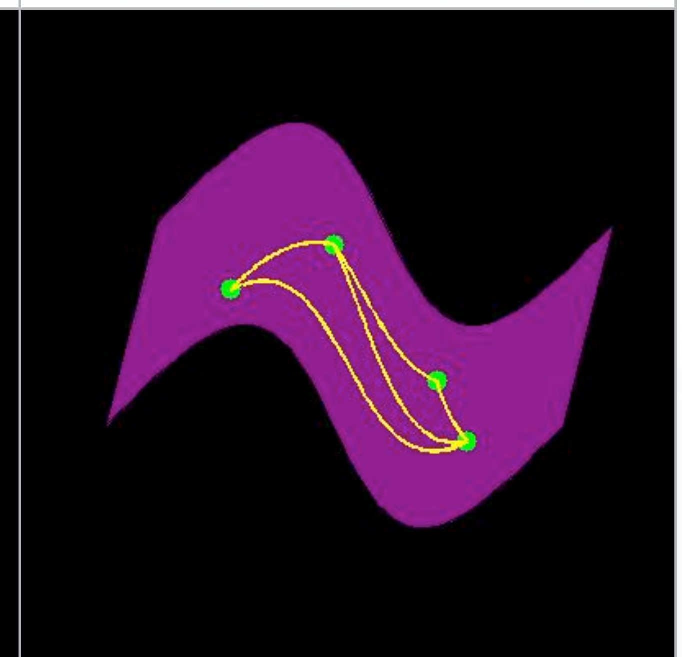
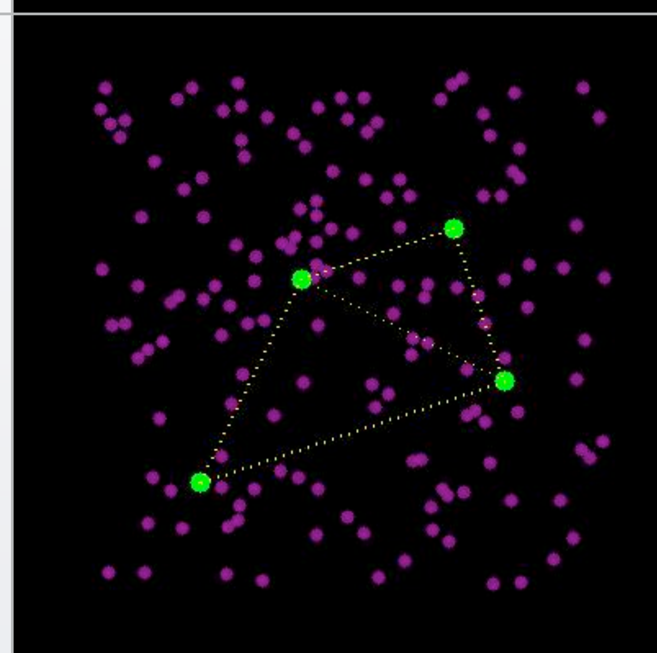
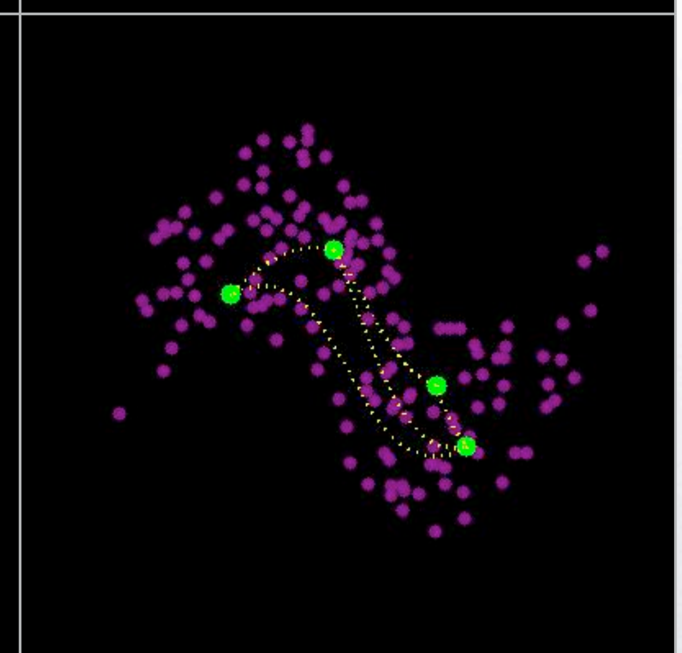
$$\sigma = [a_0, \ldots, a_k] \in \mathrm{Rips}(X, \epsilon) \Leftrightarrow |a_i - a_j| \leq \epsilon, \forall i, j$$

too many vertices

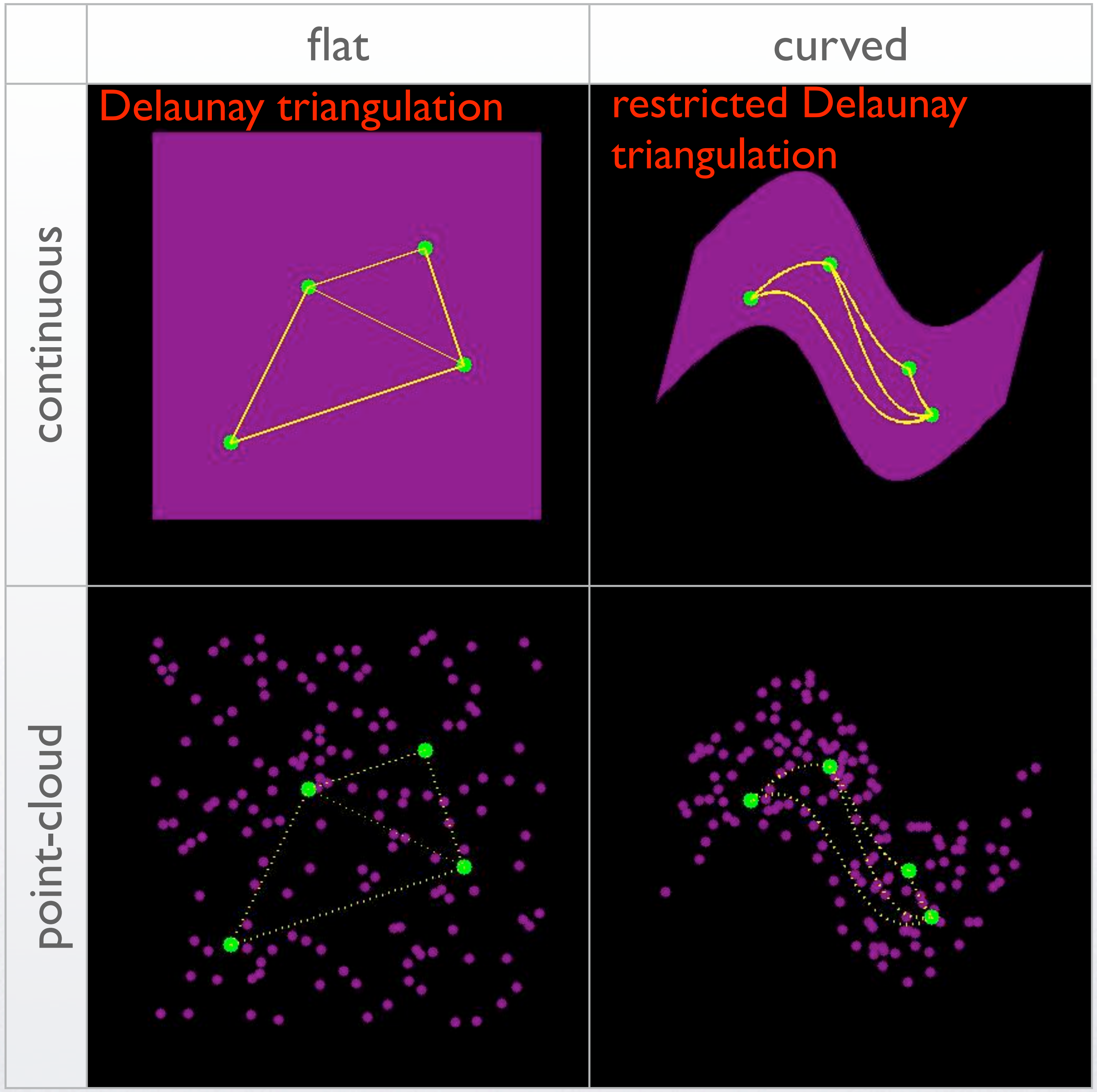
Witness complex paradigm(I)



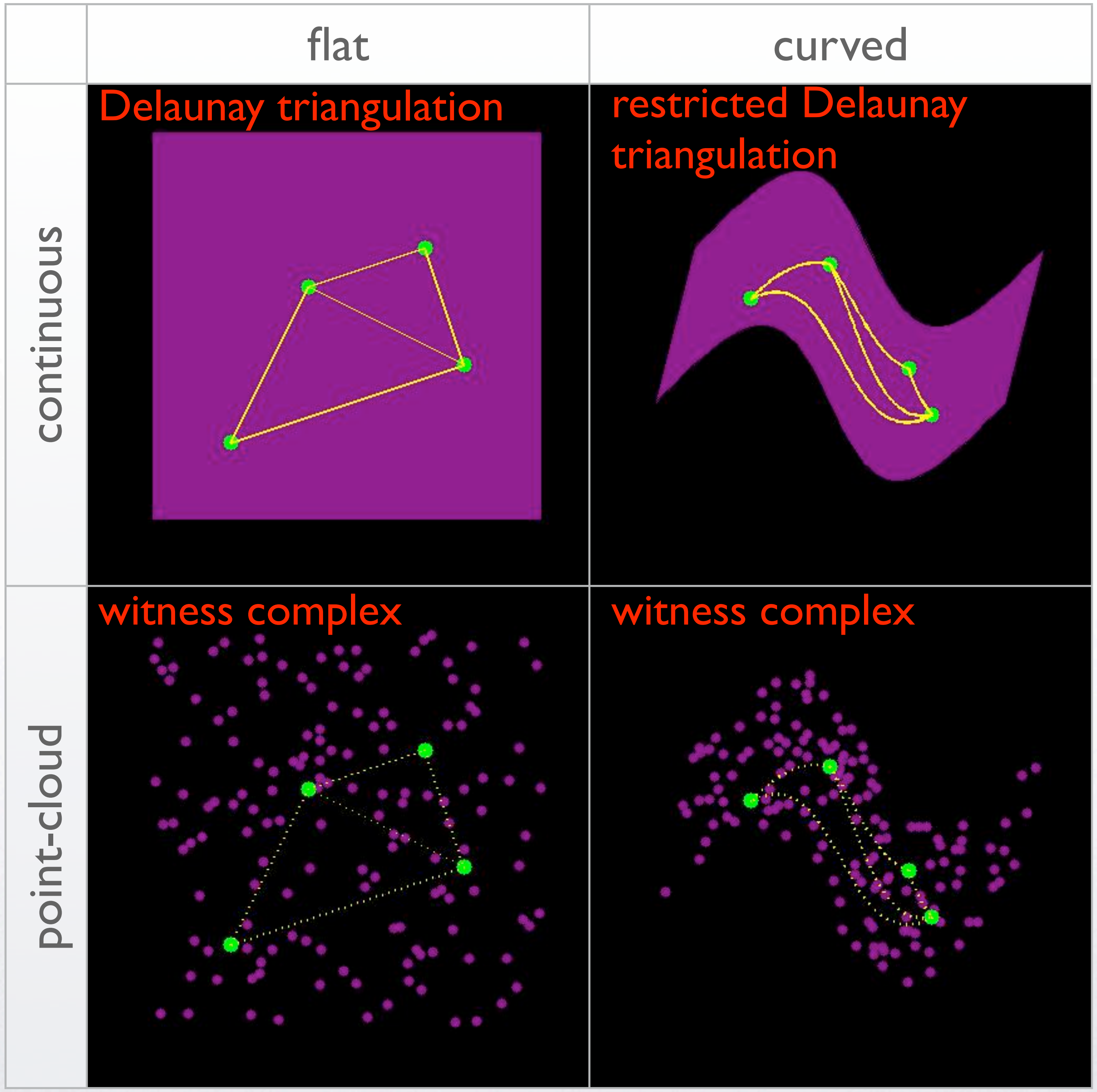
Witness complex paradigm(1)

	flat	curved
continuous		
point-cloud		

Witness complex paradigm(1)



Witness complex paradigm(1)



Vin de Silva
http://pages.pomona.edu/~vds04747/

Witness complexes

- A, X subsets of a metric space
- Strong witnesses

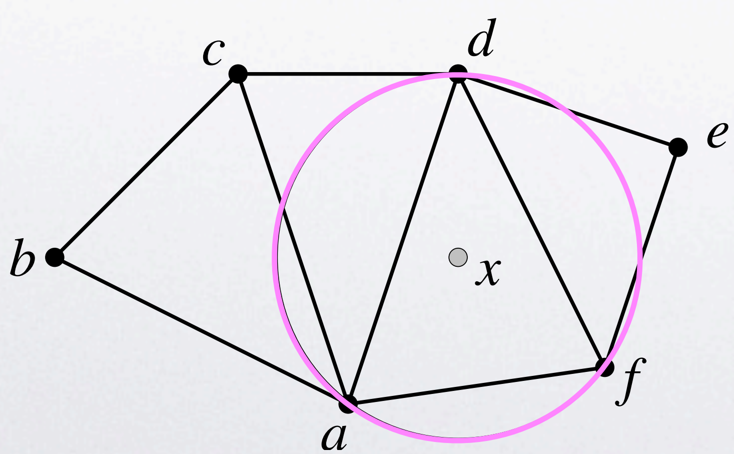
$x \in X$ is a strong witness for $\sigma \subset A$

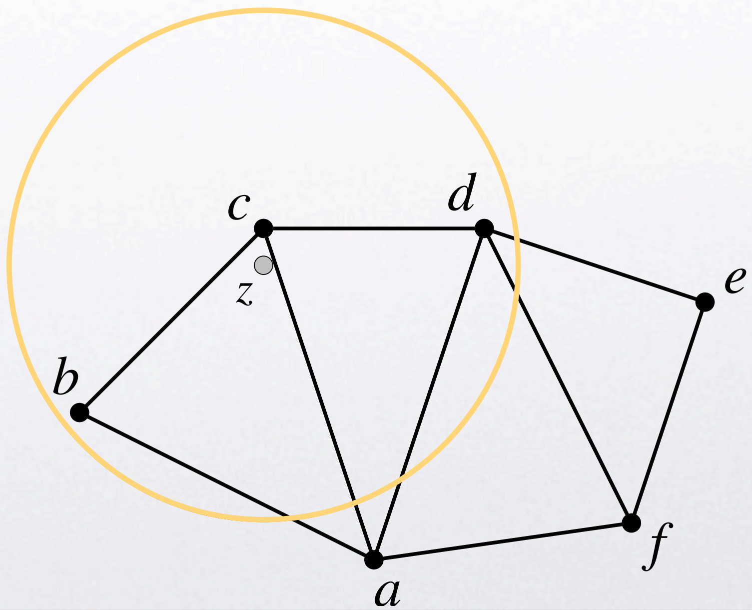
$\Leftrightarrow \quad |x-a| \leq |x-b| \quad \text{for all } a \in \sigma,\ b \in A$

- Weak witnesses

$x \in X$ is a weak witness for $\sigma \subset A$

$\Leftrightarrow \quad |x-a| \leq |x-b| \quad \text{for all } a \in \sigma,\ b \in A \setminus \sigma$





Witness complexes

- A, X subsets of a metric space
- Strong witnesses

$x \in X$ is a strong witness for $\sigma \subset A$

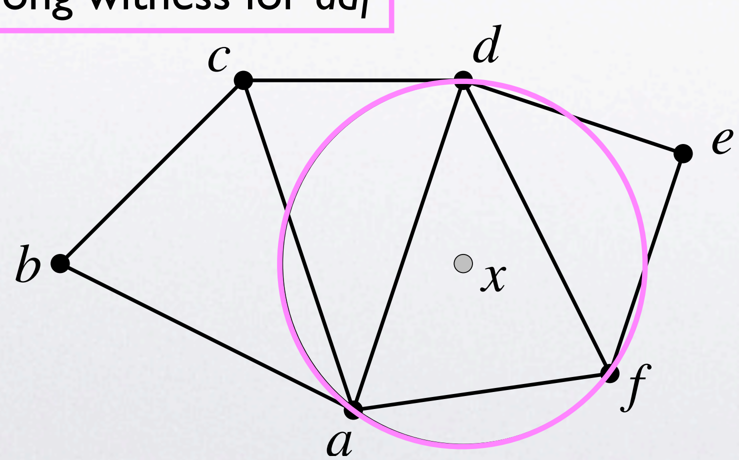
$\Leftrightarrow \quad |x - a| \leq |x - b| \quad$ for all $a \in \sigma,\ b \in A$

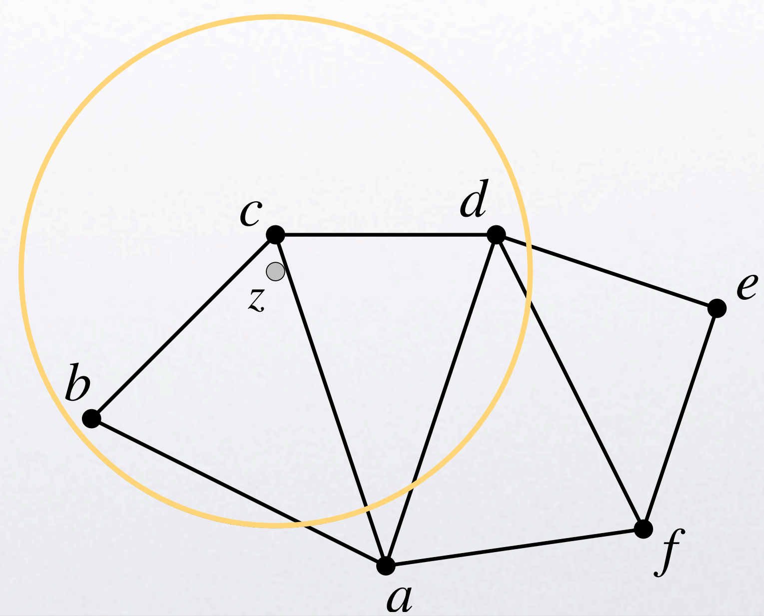
- Weak witnesses

$x \in X$ is a weak witness for $\sigma \subset A$

$\Leftrightarrow \quad |x - a| \leq |x - b| \quad$ for all $a \in \sigma,\ b \in A \setminus \sigma$

x is a strong witness for *adf*





Witness complexes

- A, X subsets of a metric space
- Strong witnesses

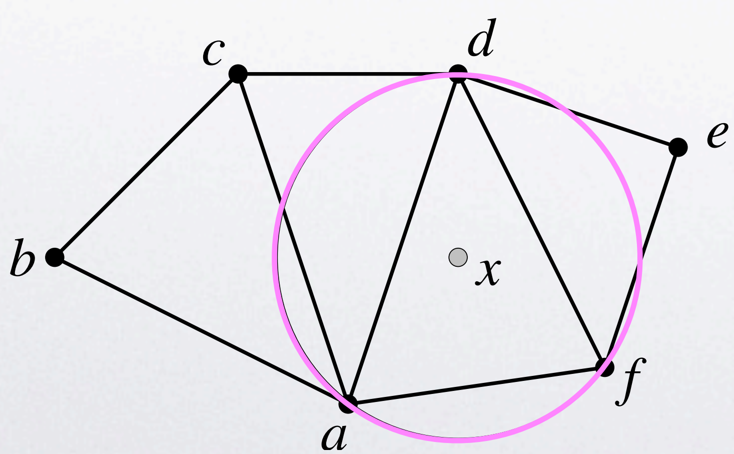
  $x \in X$ is a strong witness for $\sigma \subset A$

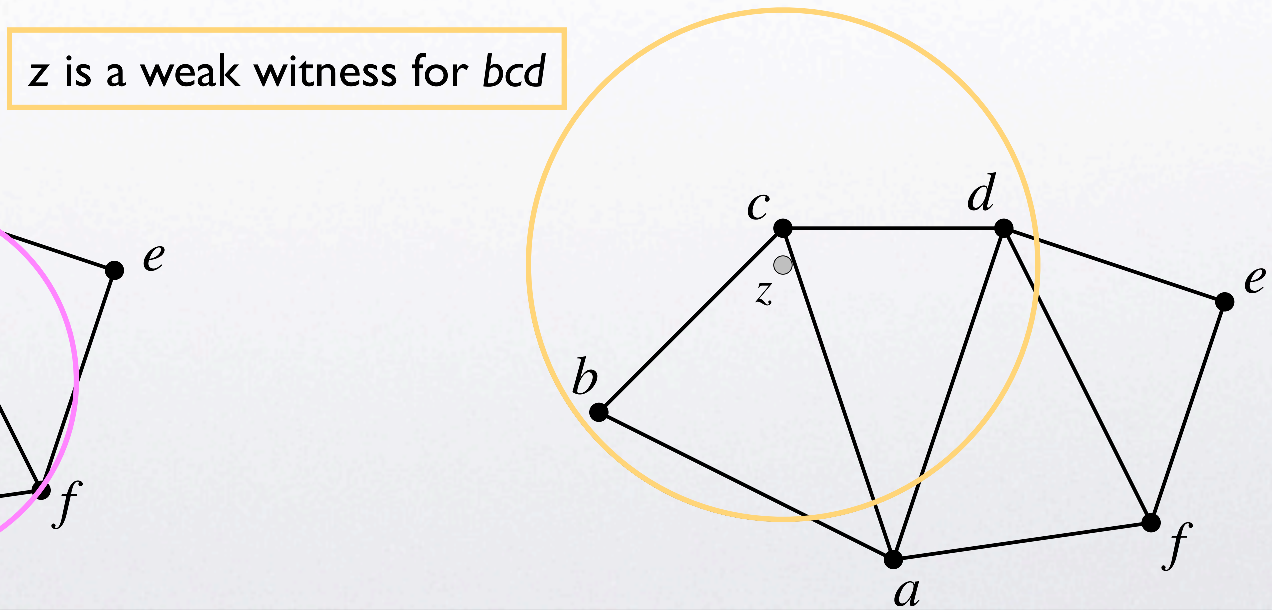
  $\Leftrightarrow \quad |x-a| \leq |x-b| \quad \text{for all } a \in \sigma,\ b \in A$

- Weak witnesses

  $x \in X$ is a weak witness for $\sigma \subset A$

  $\Leftrightarrow \quad |x-a| \leq |x-b| \quad \text{for all } a \in \sigma,\ b \in A \setminus \sigma$





Witness complexes

- A, X subsets of a metric space
- Strong witnesses

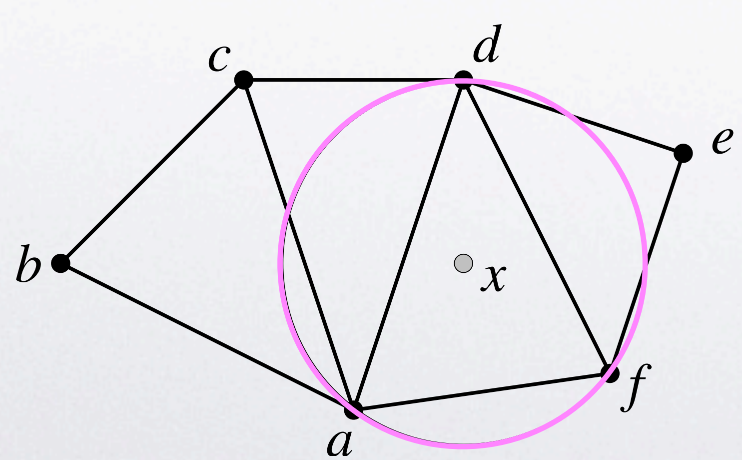
$x \in X$ is a strong witness for $\sigma \subset A$

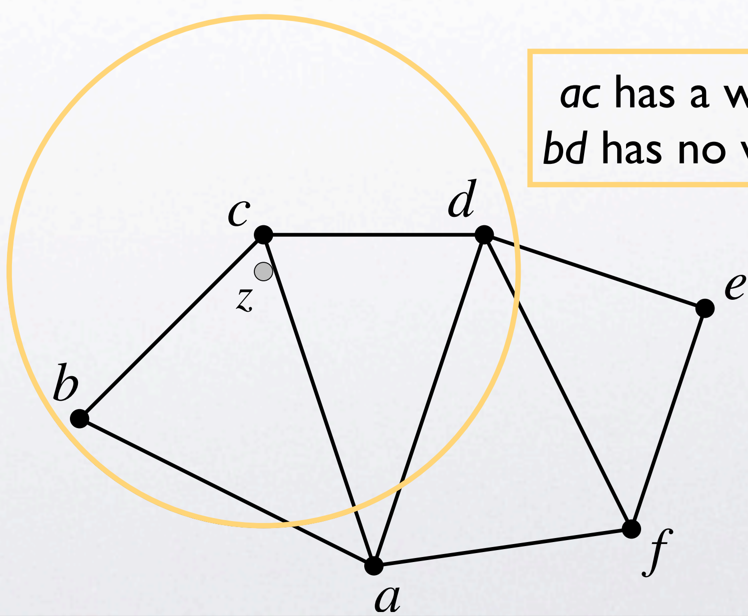
$\Leftrightarrow \quad |x - a| \leq |x - b| \quad \text{for all } a \in \sigma,\ b \in A$

- Weak witnesses

$x \in X$ is a weak witness for $\sigma \subset A$

$\Leftrightarrow \quad |x - a| \leq |x - b| \quad \text{for all } a \in \sigma,\ b \in A \setminus \sigma$





ac has a weak witness
bd has no weak witness

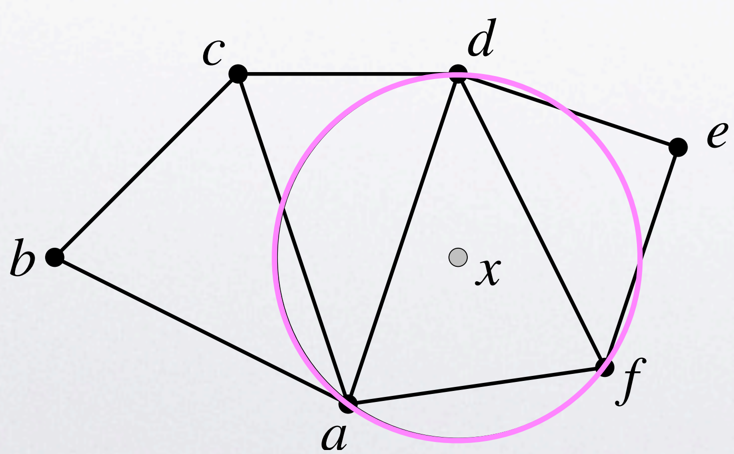
Witness complexes

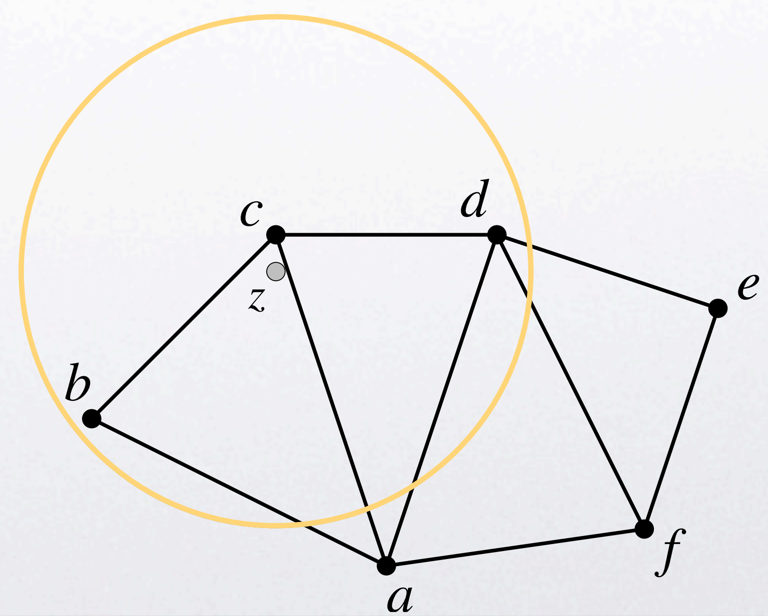
- A, X subsets of a metric space
- Strong witnesses

  $x \in X$ is a strong witness for $\sigma \subset A$
  $\Leftrightarrow \quad |x - a| \leq |x - b| \quad$ for all $a \in \sigma,\ b \in A$

- Weak witnesses

  $x \in X$ is a weak witness for $\sigma \subset A$
  $\Leftrightarrow \quad |x - a| \leq |x - b| \quad$ for all $a \in \sigma,\ b \in A \setminus \sigma$





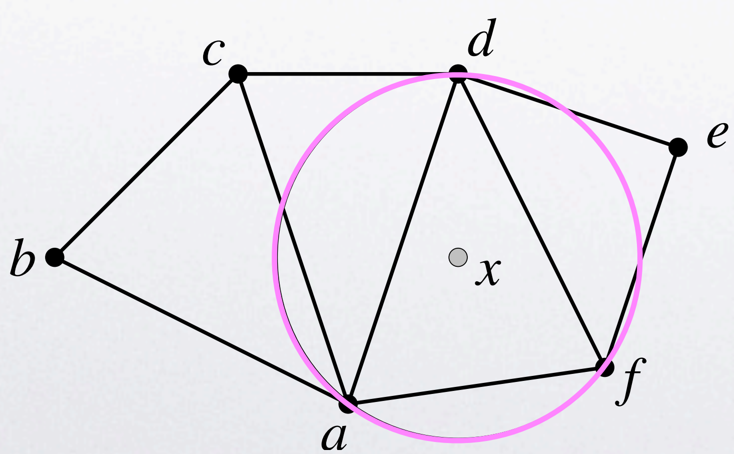
Witness complexes

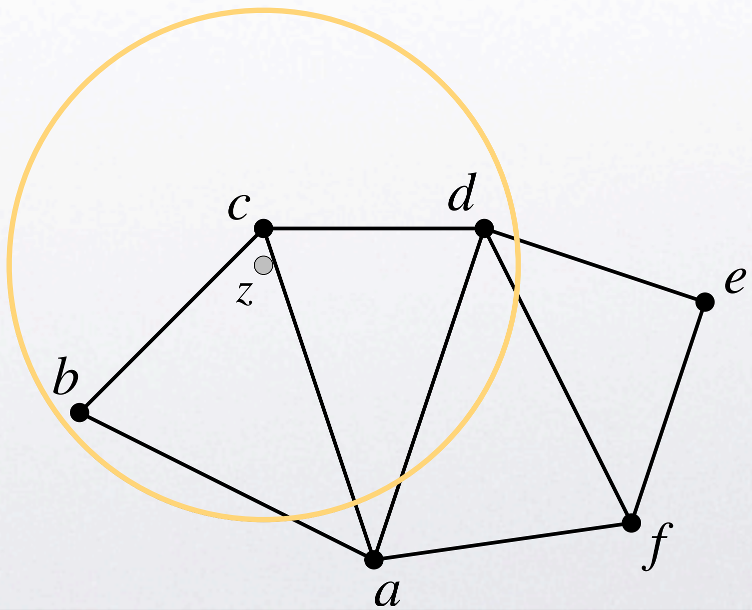
- A, X subsets of a metric space
- Strong Delaunay complex

$$\sigma \in \mathrm{Del}(A, X) \quad \Leftrightarrow \quad \sigma \text{ has a strong witness } x \in X$$

- Weak Delaunay complex

$$\sigma \in \mathrm{Del}^{\mathrm{w}}(A, X) \quad \Leftrightarrow \quad \text{every } \tau \leq \sigma \text{ has a weak witness } x \in X$$





The weak witnesses theorem

$$\mathrm{Del}(A,\mathbf{R}^n) = \mathrm{Del}^w(A,\mathbf{R}^n)$$

$S \subseteq A$ has a strong witness

$\Leftrightarrow$

every $T \subseteq S$ has a weak witness

$\Rightarrow$ trivial

$\Leftarrow$ construct strong witness in convex hull of weak witnesses

Witness complexes

- For $A \subset \mathbf{R}^n$
  - $\mathrm{Del}(A,\mathbf{R}^n)$ = Delaunay triangulation
  - $\mathrm{Del}^w(A,\mathbf{R}^n) = \mathrm{Del}(A,\mathbf{R}^n)$ weak witnesses theorem
- For $A \subset X \subset \mathbf{R}^n$
  - $\mathrm{Del}(A,X)$ = restricted Delaunay triangulation
- If $B \subset \mathbf{R}^n$ discrete, choose landmark set $A \subset B$
  - $\mathrm{Del}(A,B)$ is called a strong witness complex for B
  - $\mathrm{Del}^w(A,B)$ is called a weak witness complex for B

Witness complex paradigm(2)

X —— Čech nerve theorem —— Del(A,X) —— weak witnesses theorem —— $\mathrm{Del}^w(A,X)$

Del(A,X) | almost surely unequal | Del(A,B)

$\mathrm{Del}^w(A,X)$ | $\mathrm{Del}^w(A,B)$

Green means "plausibly equal"

Red means "clearly unequal"

Witness complex paradigm(2)

X —— Čech nerve theorem —— Del(A,X) —— weak witnesses theorem —— Del^w(A,X)

Del(A,X) | almost surely unequal | Del(A,B)

Del^w(A,X) | Del^w(A,B)

Strong witnesses exist with probability zero ⟶ Del(A,B)

Green means "plausibly equal"

Red means "clearly unequal"

Witness complex paradigm(2)

X —— Čech nerve theorem —— Del(A,X) —— weak witnesses theorem —— $\mathrm{Del}^w(A,X)$

Del(A,X) | almost surely unequal | Del(A,B)

$\mathrm{Del}^w(A,X)$ | $\mathrm{Del}^w(A,B)$

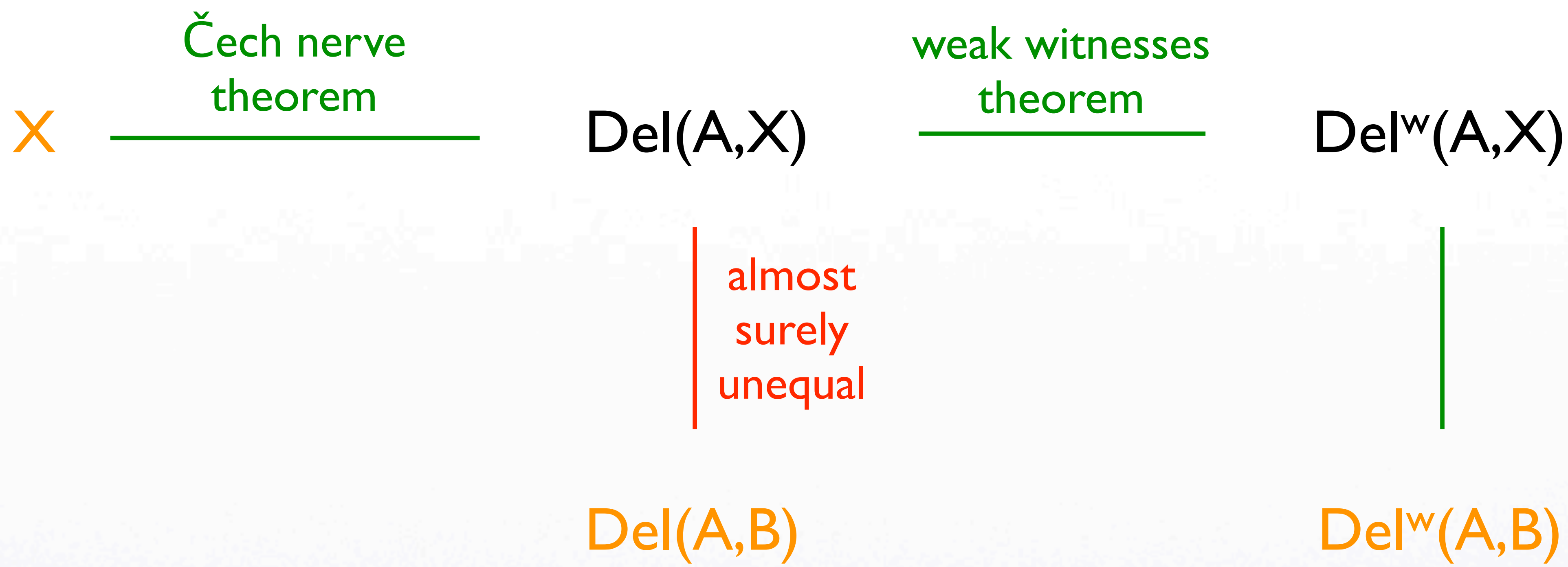
Strong witnesses exist with probability zero → Del(A,B)

beware of slivers → $\mathrm{Del}^w(A,B)$

Green means "plausibly equal"

Red means "clearly unequal"

Witness complex paradigm(2)



- Recent theoretical work on X = Del^w(A,B)
  - Attali, Edelsbrunner, Mileyko curves, surfaces
  - Boissonat, Guibas, Oudot sliver exudation, thickened witnesses

The weak witnesses theorem

$$\mathrm{Del}(A,\mathbf{R}^n) = \mathrm{Del}^w(A,\mathbf{R}^n)$$

$S \subseteq A$ has a strong witness

$\Leftrightarrow$

every $T \subseteq S$ has a weak witness

$\Rightarrow$ trivial

$\Leftarrow$ construct strong witness in convex hull of weak witnesses

Voronoi convexity

Under the following assumptions...

- topological space X, set A
- $\forall x,y \in X$, $\exists$ connected $\gamma(x,y) \ni x,y$
- $\forall a \in A$, $\exists$ continuous function $d(a,x)$
- $\forall x,y \in X$, the Voronoi half-space

$$R(a,b) = \{x \in X \mid d(a,x) \leq d(b,x)\}$$

is γ-convex (i.e. closed under γ)

...it follows that $Del(A,X) = Del^w(A,X)$

Voronoi convexity

Under the following assumptions...

- topological space X, set A
- $\forall x,y \in X$, $\exists$ connected $\gamma(x,y) \ni x,y$ — convex hull of {x,y}
- $\forall a \in A$, $\exists$ continuous function d(a,x)
- $\forall x,y \in X$, the Voronoi half-space

  $$R(a,b) = \{x \in X \mid d(a,x) \leq d(b,x)\}$$

  is γ-convex (i.e. closed under γ)

...it follows that $Del(A,X) = Del^w(A,X)$

Examples

- Voronoi convexity is satisfied by:
  - $A \subset X = \mathbf{R}^n$, $d(a,x) = |a-x|$ = geodesic metric
  - $A \subset X = \frac{1}{2}\mathbf{S}^n$ (hemisphere), $d(a,x)$ = geodesic metric
  - $A \subset X = \mathbf{H}^n$ (hyperbolic space), $d(a,x)$ = geodesic metric
  - $A \subset X = \mathbf{T}$ (tree), $d(a,x)$ = geodesic metric
  - $A \subset X = \mathbf{R}^{p,q}$, $d(a,x) = |a-x|^2$ = Minkowski square norm
  - $A \subset X = \mathbf{R}^{p,q}$, $d(a,x) = a*x$ = Minkowski inner product
  - $X = \mathbf{R}^n, A \subset \mathbf{R}^n \times \mathbf{R}$, $d((a,c),x) = a.x - c$ (linear inequalities)

Laguerre diagrams

Weight schemes satisfying Voronoi convexity

Euclidean

$$D(a, w(a), x) = \tfrac{1}{2}|a-x|^2 - w(a)$$

Spherical (restrict to hemisphere)

$$D(a, w(a), x) = -e^{w(a)} \cos(\theta(a,x))$$

Hyperbolic

$$D(a, w(a), x) = e^{-w(a)} \cosh(u(a,x))$$

Inequalities $D(a,w(a),x) \leq D(b,w(b),x)$ define half-spaces

(Spherical Laguerre diagrams due to Sugihara, 2002)

Tolerance

- $x \in \mathbf{R}^n$ is a strong ϵ-witness for $S \subseteq A$

  $\Leftrightarrow |a-x|^2 \leq |b-x|^2+\epsilon$, for all $a \in S$, $b \in A$

- $x \in \mathbf{R}^n$ is a weak ϵ-witness for $S \subseteq A$

  $\Leftrightarrow |a-x|^2 \leq |b-x|^2+\epsilon$, for all $a \in S$, $b \in A - S$

Tolerance

- x is a strong ϵ-witness for $S \subseteq A$

  $\Leftrightarrow D(x,\epsilon,a) \leq D(x,0,b)$, for all $a \in S$, $b \in A$

- x is a weak ϵ-witness for $S \subseteq A$

  $\Leftrightarrow D(x,\epsilon,a) \leq D(x,0,b)$, for all $a \in S$, $b \in A\text{-}S$

Tolerance

Laguerre weights in Euclidean, spherical, hyperbolic spaces

- x is a strong ϵ-witness for $S \subseteq A$

  $\Leftrightarrow D(x,\epsilon,a) \leq D(x,0,b)$, for all $a \in S$, $b \in A$

- x is a weak ϵ-witness for $S \subseteq A$

  $\Leftrightarrow D(x,\epsilon,a) \leq D(x,0,b)$, for all $a \in S$, $b \in A\text{-}S$

ϵ-witness compexes

- strong ϵ-witness complex

  $S \in \mathrm{Del}(A,X;\epsilon) \Leftrightarrow$ S has a strong ϵ-witness in X

- weak ϵ-witness complex

  $S \in \mathrm{Del}^w(A,X;\epsilon) \Leftrightarrow$ every $T \subseteq S$ has a weak ϵ-witness in X

Vin de Silva
http://pages.pomona.edu/~vds04747/

ϵ-witness compexes

Filtered complexes for persistent homology

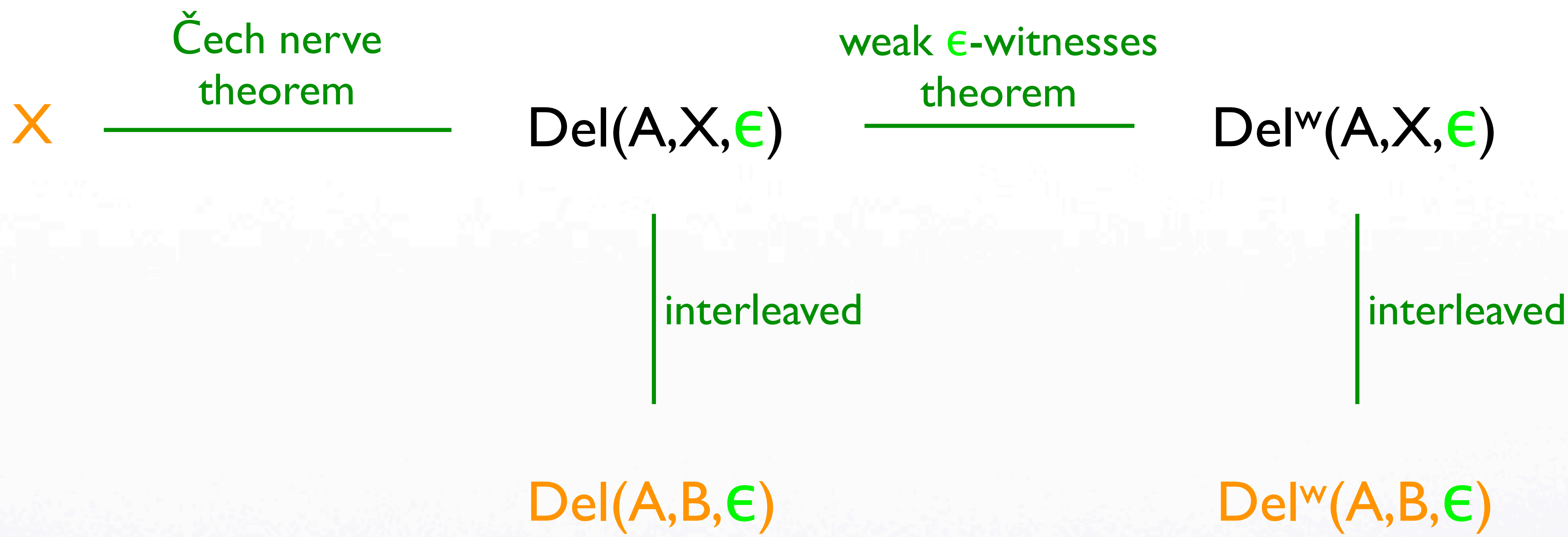
- strong ϵ-witness complex

  $S \in \mathrm{Del}(A,X;\epsilon) \Leftrightarrow$ S has a strong ϵ-witness in X

- weak ϵ-witness complex

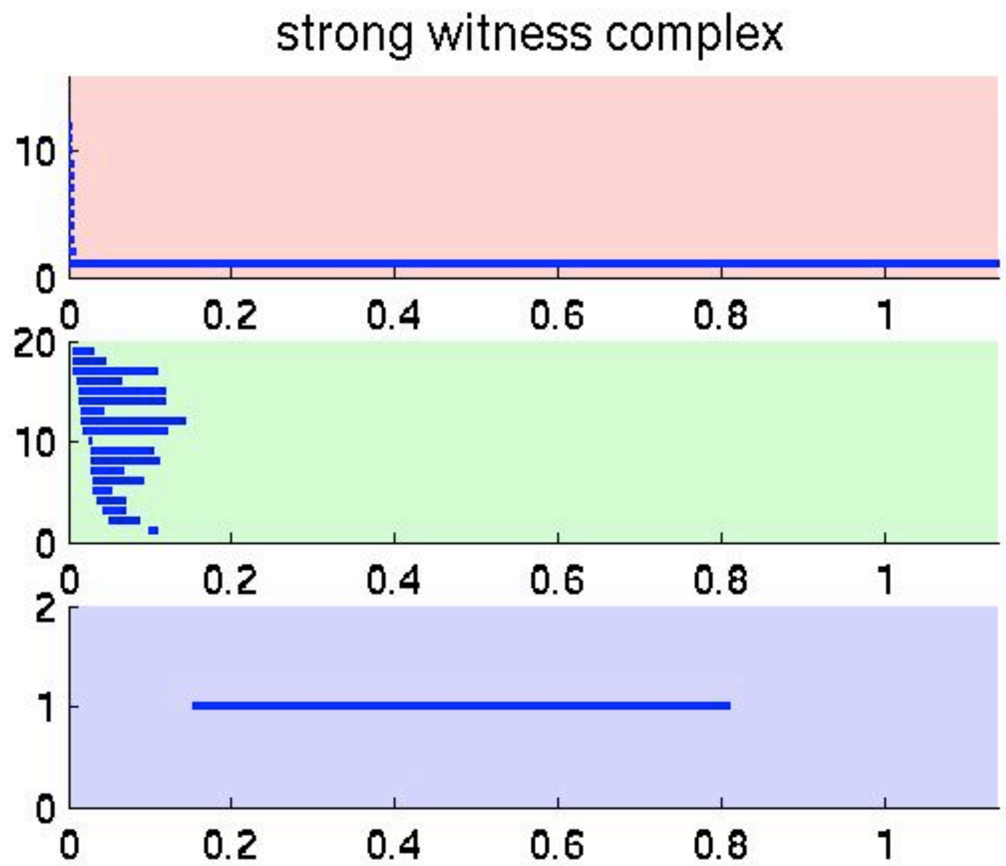
  $S \in \mathrm{Del}^w(A,X;\epsilon) \Leftrightarrow$ every $T \subseteq S$ has a weak ϵ-witness in X

Witness complex paradigm(ϵ)



Green means "plausibly equal"

Comparing strong and weak



Data points sampled from 2-sphere

And the Oscar goes to...

Witness Complexes -- Mumford Dataset

Vin de Silva & Gunnar Carlsson

Topology in the 21st Century

Vin de Silva
Pomona College

Discrete Laplacians

Standard Pipeline (second attempt)

hidden/unknown space X

↓

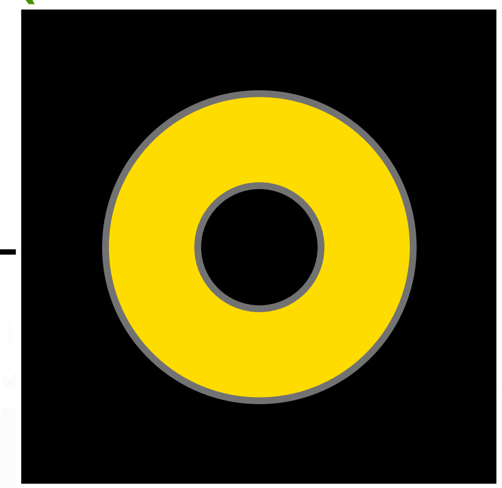
finite sample $Y \subset X$

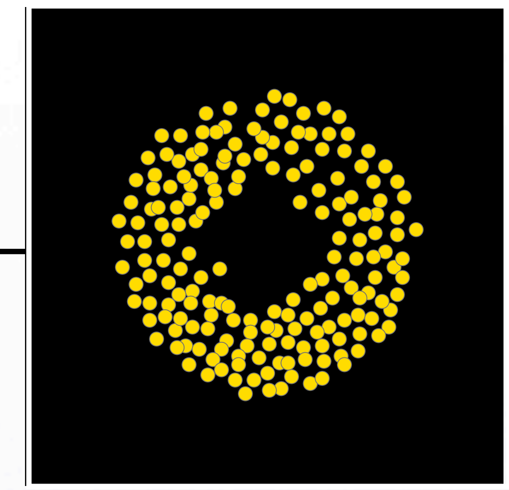
↓

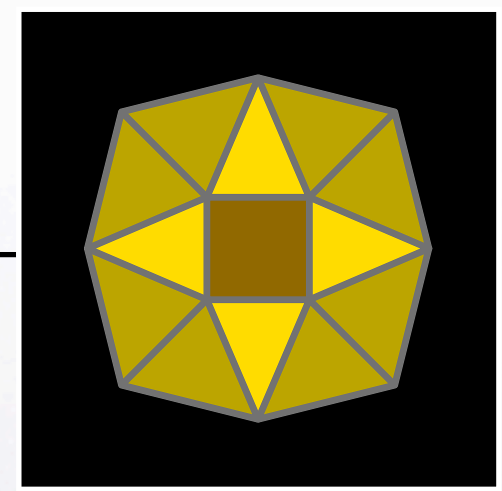
labelled complex $S(r) = S(Y,r)$

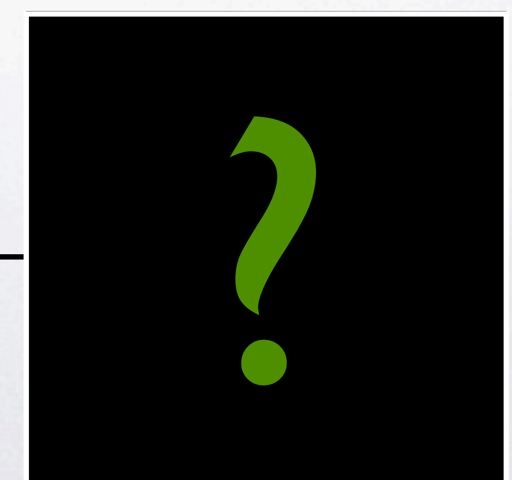
↓

quantitative topology









Vin de Silva
http://pages.pomona.edu/~vds04747/

The discrete Laplacian Δ_k

- C_k = { real-valued functions on k-simplices of S(Y) }
  - floating point rather than exact arithmetic
- Define discrete Laplacian operators $\Delta_k : C_k \to C_k$
- Consider the harmonic spaces $H_k = \mathrm{Ker}(\Delta_k)$
  - H_k is isomorphic to standard homology of X
- Consider eigenspaces $\{ f : \Delta_k f = \lambda f \}$ for λ small
  - "almost homology" or "ϵ-homology"
- Information derived from the ranks of these spaces (Betti numbers) and the eigenfunctions themselves

Constructing Δ_k

Given a chain complex over the real numbers...

$$\cdots C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_k} C_{k+1} \cdots$$

homology is defined using a chain complex

...and an inner product on each C_k, we can form the dual cochain complex:

$$\cdots C_{k-1} \xrightarrow{\partial_k^*} C_k \xrightarrow{\partial_k^*} C_{k+1} \cdots$$

cohomology is defined using a cochain complex

The discrete Laplacian is defined...

$$\Delta_k = \partial_k^* \partial_k + \partial_{k+1} \partial_{k+1}^*$$

...and one can easily prove (in the finite dimensional case):

harmonic space $$\mathcal{H}_k := \operatorname{Ker}(\Delta_k) \cong \frac{\operatorname{Ker}(\partial_k)}{\operatorname{Im}(\partial_{k+1})} =: H_k$$ homology

Aside: Hodge theory

For a 3-dimensional domain:

$$\Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla\times} \Omega^2 \xrightarrow{\nabla\cdot} \Omega^3$$

$$\Omega^0 \xleftarrow{-\nabla\cdot} \Omega^1 \xleftarrow{\nabla\times} \Omega^2 \xleftarrow{-\nabla} \Omega^3$$

For example:

$$\begin{aligned} \Delta_0 f &:= \nabla \cdot (\nabla f) &= -\textstyle\sum_{i=1}^3 \frac{\partial^2 f}{\partial x_i^2} \\ \Delta_0 \vec{g} &:= \nabla \times (\nabla \times \vec{g}) - \nabla(\nabla \cdot \vec{g}) &= -\textstyle\sum_{i=1}^3 \frac{\partial^2 \vec{g}}{\partial x_i^2} \end{aligned}$$

Proof that Ker(Δ_k) = H_k is much more difficult in this setting.

Aside: Hodge theory

For a 3-dimensional domain:

$$\underset{\text{scalar fields}}{\Omega^0} \xrightarrow[\text{grad}]{\nabla} \underset{\text{vector fields}}{\Omega^1} \xrightarrow[\text{curl}]{\nabla\times} \underset{\text{vector fields}}{\Omega^2} \xrightarrow[\text{div}]{\nabla\cdot} \underset{\text{scalar fields}}{\Omega^3}$$

$$\Omega^0 \xleftarrow{-\nabla\cdot} \Omega^1 \xleftarrow{\nabla\times} \Omega^2 \xleftarrow{-\nabla} \Omega^3$$

For example:

$$\Delta_0 f \quad := \quad \nabla \cdot (\nabla f) \quad = \quad -\sum_{i=1}^3 \frac{\partial^2 f}{\partial x_i^2}$$

$$\Delta_0 \vec{g} \quad := \quad \nabla \times (\nabla \times \vec{g}) - \nabla(\nabla \cdot \vec{g}) \quad = \quad -\sum_{i=1}^3 \frac{\partial^2 \vec{g}}{\partial x_i^2}$$

Proof that $\mathrm{Ker}(\Delta_k) = H_k$ is much more difficult in this setting.

Aside: Hodge theory

For a 3-dimensional domain:

$$\underset{\text{scalar fields}}{\Omega^0} \xrightarrow[\text{grad}]{\nabla} \underset{\text{vector fields}}{\Omega^1} \xrightarrow[\text{curl}]{\nabla\times} \underset{\text{vector fields}}{\Omega^2} \xrightarrow[\text{div}]{\nabla\cdot} \underset{\text{scalar fields}}{\Omega^3}$$

$$\Omega^0 \xleftarrow{-\nabla\cdot} \Omega^1 \xleftarrow{\nabla\times} \Omega^2 \xleftarrow{-\nabla} \Omega^3$$

For example:

$$\Delta_0 f := \nabla\cdot(\nabla f) = -\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}$$

$$\Delta_0 \vec{g} := \nabla\times(\nabla\times\vec{g}) - \nabla(\nabla\cdot\vec{g}) = -\sum_{i=1}^{3} \frac{\partial^2 \vec{g}}{\partial x_i^2}$$

Proof that Ker(Δ_k) = H_k is much more difficult in this setting.

requires analysis in spaces of smooth functions for lemmas of the form Ker(D)=Im(D*)$^{\perp}$

Aside: Hodge theory

For a 3-dimensional domain:

$$\underset{\text{fields}}{\overset{\text{scalar}}{\Omega^0}} \xrightarrow[\text{grad}]{\nabla} \underset{\text{fields}}{\overset{\text{vector}}{\Omega^1}} \xrightarrow[\text{curl}]{\nabla\times} \underset{\text{fields}}{\overset{\text{vector}}{\Omega^2}} \xrightarrow[\text{div}]{\nabla\cdot} \underset{\text{fields}}{\overset{\text{scalar}}{\Omega^3}}$$

$$\Omega^0 \xleftarrow{-\nabla\cdot} \Omega^1 \xleftarrow{\nabla\times} \Omega^2 \xleftarrow{-\nabla} \Omega^3$$

For example:

$$\Delta_0 f := \nabla\cdot(\nabla f) = -\sum_{i=1}^3 \frac{\partial^2 f}{\partial x_i^2}$$

$$\Delta_0 \vec{g} := \nabla\times(\nabla\times\vec{g}) - \nabla(\nabla\cdot\vec{g}) = -\sum_{i=1}^3 \frac{\partial^2 \vec{g}}{\partial x_i^2}$$

Proof that $\mathrm{Ker}(\Delta_k) = H_k$ is much more difficult in this setting.

requires analysis in spaces of smooth functions for lemmas of the form $\mathrm{Ker}(D)=\mathrm{Im}(D^*)^\perp$

result not true for subsets of Euclidean space! only for closed manifolds

ϵ-Betti numbers

For every nonnegative integer k, and $\epsilon > 0$:

Integers b_k "Betti numbers"

Integers $b_{k+½}(\epsilon)$ "ϵ-Betti numbers"

such that:

$\dim(\mathrm{Ker}(\Delta_k)) = b_k$

$\dim(\mathrm{Eig}(\Delta_k, \epsilon)) = b_{k-½}(\epsilon) + b_k + b_{k+½}(\epsilon)$

space spanned by eigenfunctions with eigenvalue less than ϵ

Laplacian pipeline

hidden/unknown space X

↓

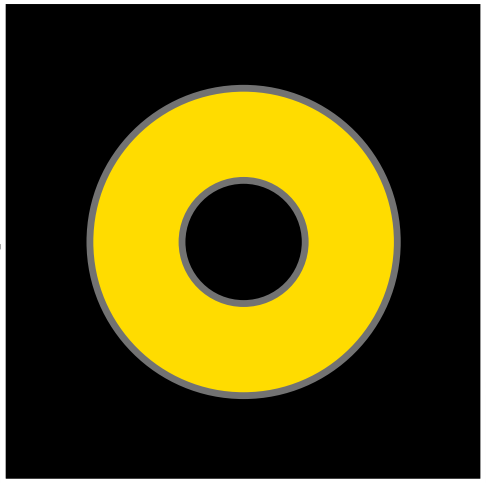
finite sample $Y \subset X$

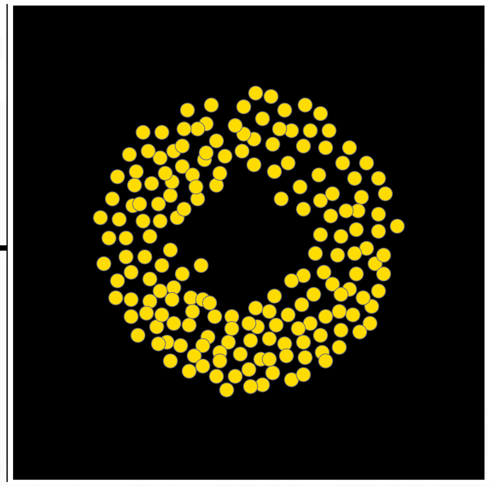
↓

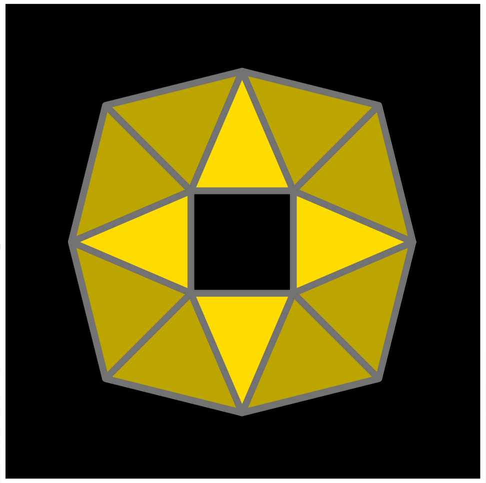
weighted complex $S = S(Y)$, $f: S \to R$

↓

ϵ-harmonic forms







$b_0 = 1$

$b_{0.5}(\epsilon) = ?$

$b_1 = 1$

$b_{1.5}(\epsilon) = ?$

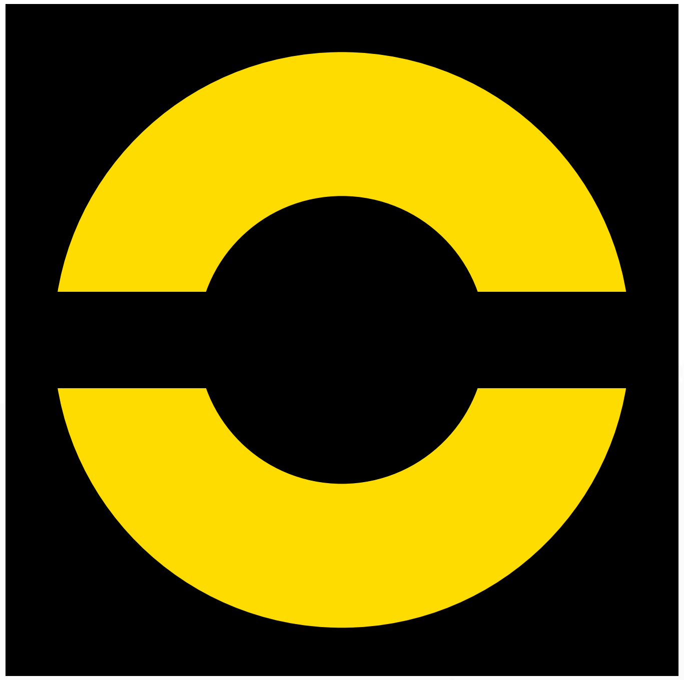
$b_2 = 0$

Pros and cons

- ✔ Several ways to incorporate continuous parameters
  - meaning of "λ is close to zero" — how close?
  - simplices can be weighted prior to construction of Δ_k
- ✔ Harmonic cycles have global optimality properties
  - localising features/minimal cycle problem
- ✔ Non-zero eigenfunctions encode subtle relationships between cells of adjacent dimensions

- ✘ Numerically more vulnerable than persistent homology
- ✘ Theory somewhat underdeveloped
- (except graph Laplacians, see "Spectral Graph Theory" by Chung ✔)
- (recent work on high-dimensional spanning "trees" by Jeremy Martin et al ✔)

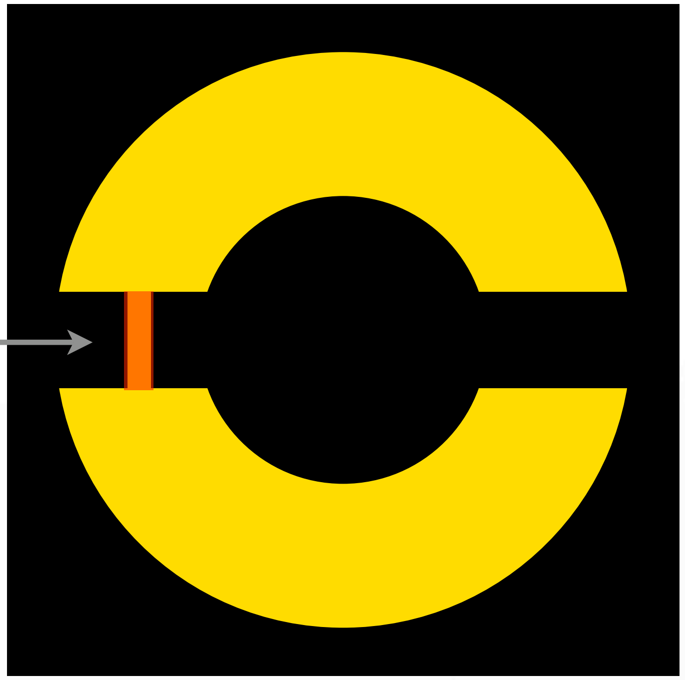
Examples



b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
2	0	0	0	0

Examples

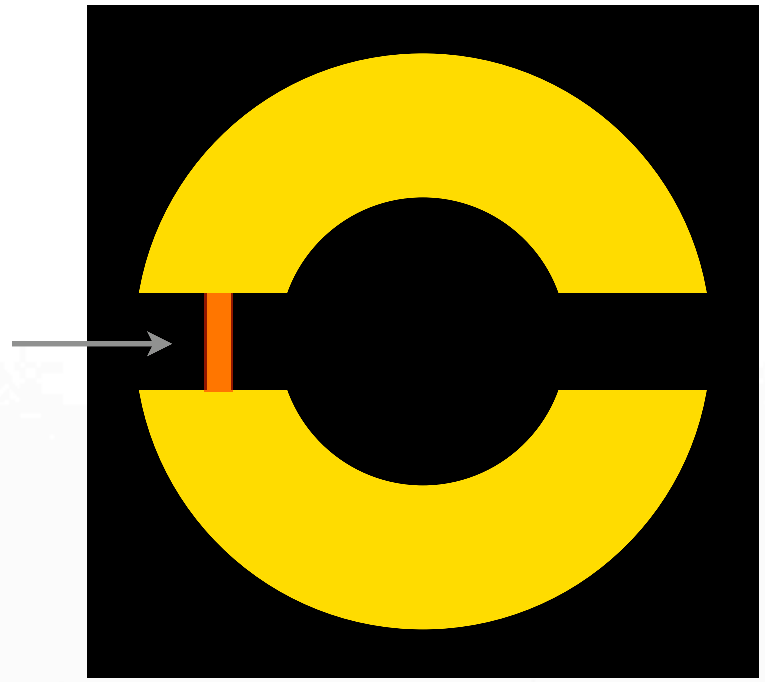
hot spot for 1-chain j, where $\Delta_1 j = \lambda j$



b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	1	0	0	0

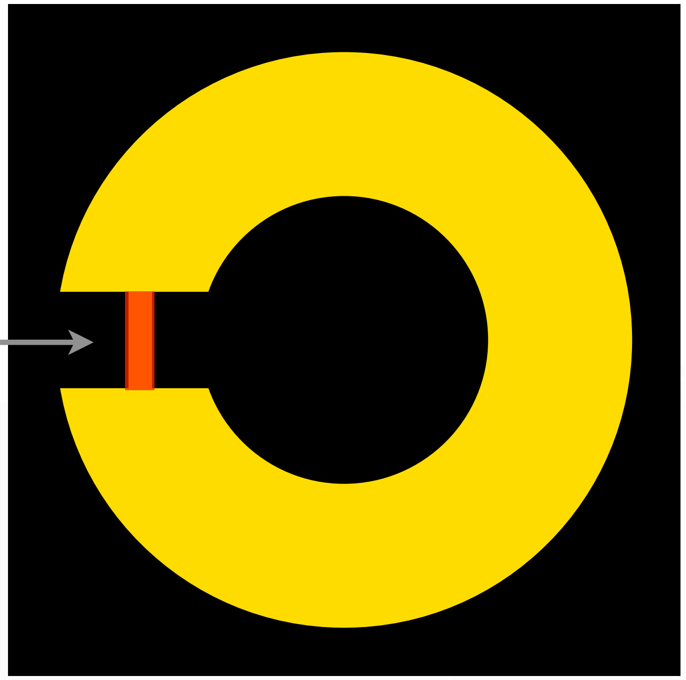
Examples

hot spot for 1-chain j, where $\Delta_1 j = \lambda j$



b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	1	0	0	0

Examples

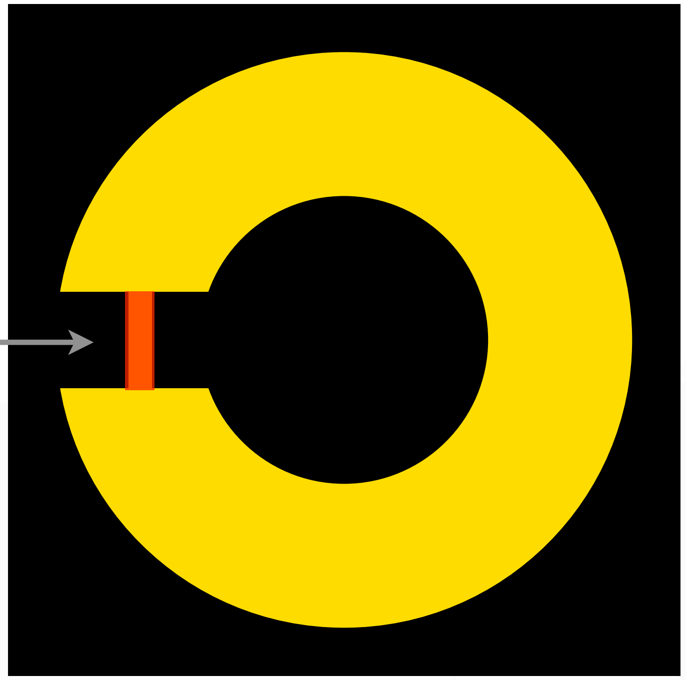


hot spot for 1-cycle j, where $\Delta_1 j = 0$

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	1	0	0

Examples

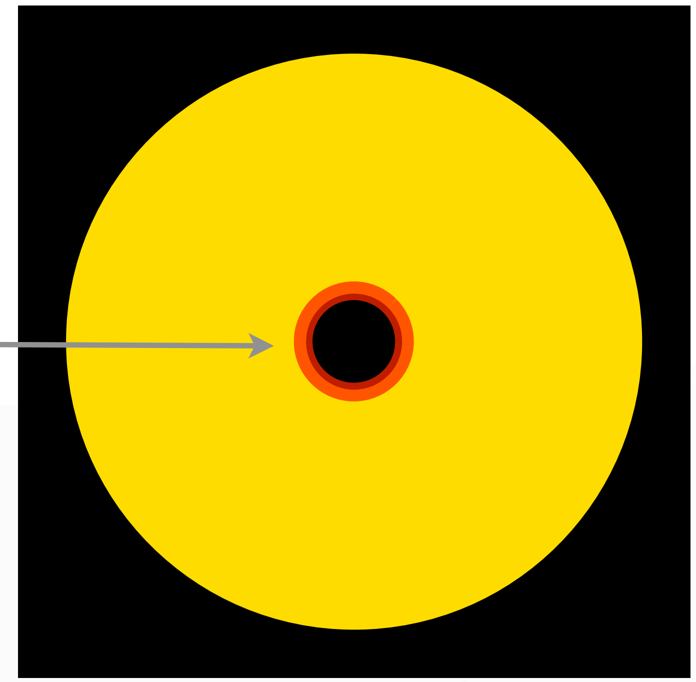
hot spot for 1-cycle j, where $\Delta_1 j = 0$



b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	1	0	0

Examples

hot spot for 1-cycle j,
where $\Delta_1 j = 0$

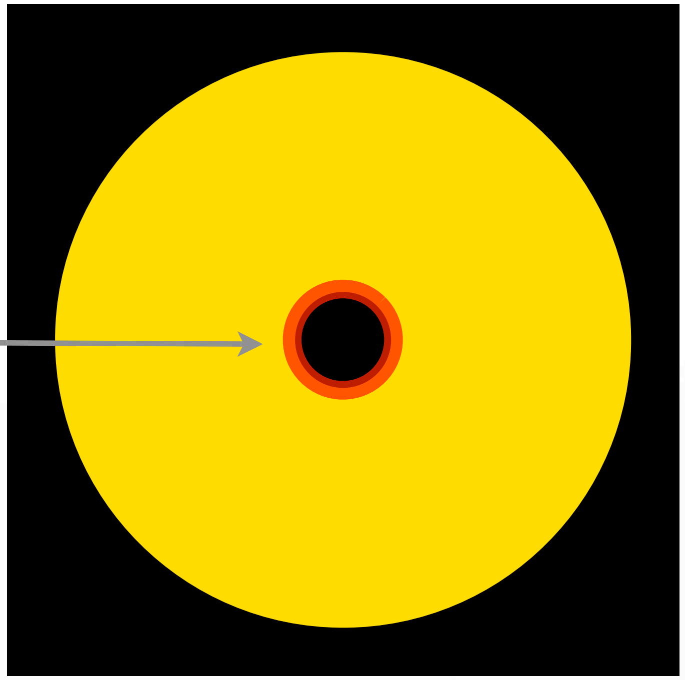


annulus

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	1	0	0

Examples

hot spot for 1-cycle j,
where $\Delta_1 j = 0$



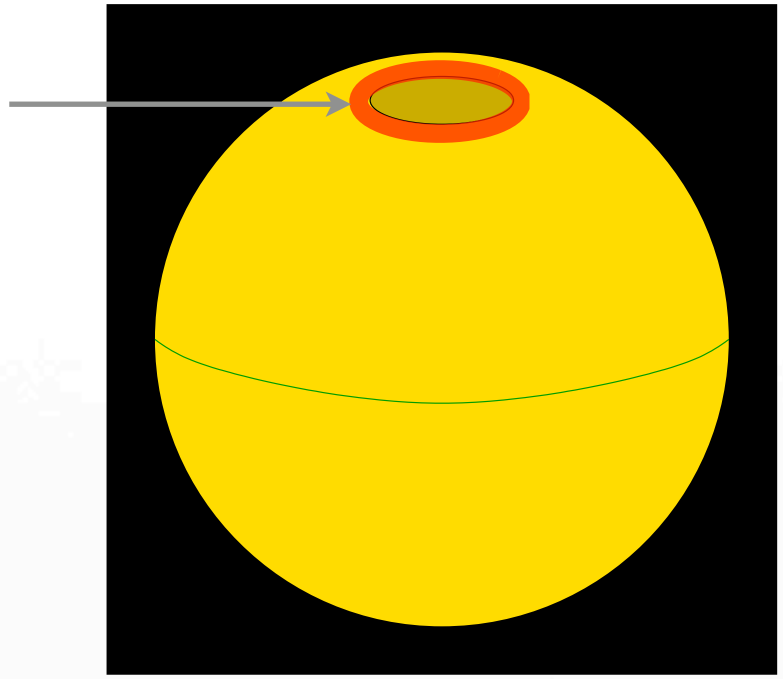
annulus

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	1	0	0

Examples

hot spot for 1-cycle j, where $\Delta_1 j = \lambda j$

hot spot for 2-chain k, where $\Delta_2 k = \lambda k$



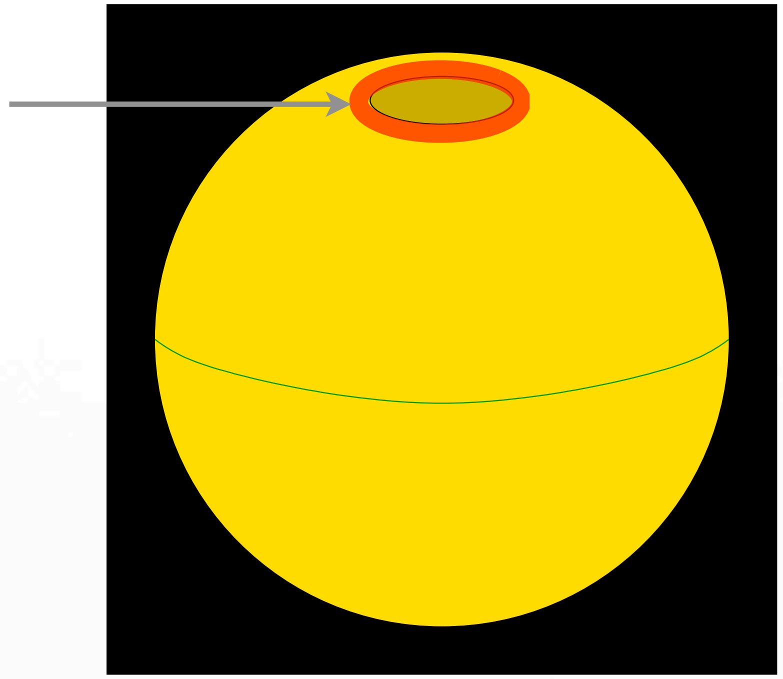
punctured sphere

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	0	1	0

hot spot for 1-cycle j, where $\Delta_1 j = \lambda j$

hot spot for 2-chain k, where $\Delta_2 k = \lambda k$

Examples



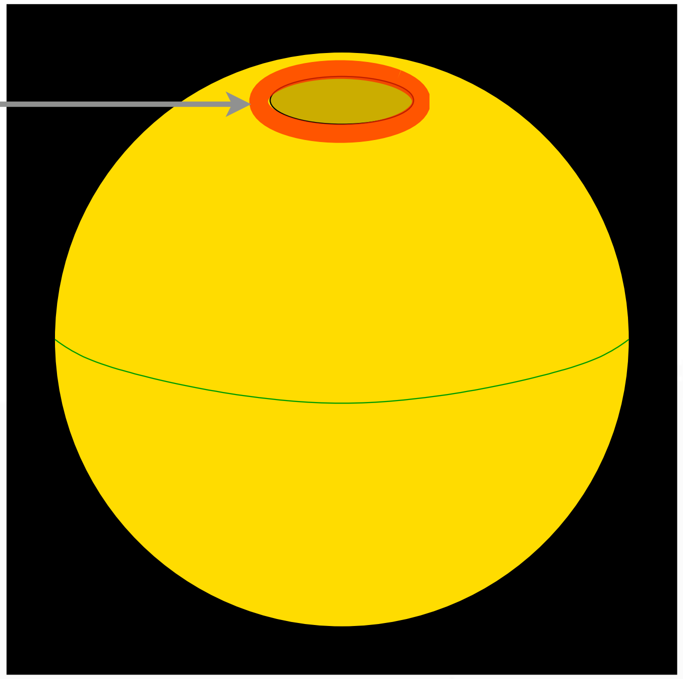
punctured sphere

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	0	1	0

hot spot for 1-cycle j, where $\Delta_1 j = \lambda j$

hot spot for 2-chain k, where $\Delta_2 k = \lambda k$

Examples



punctured sphere

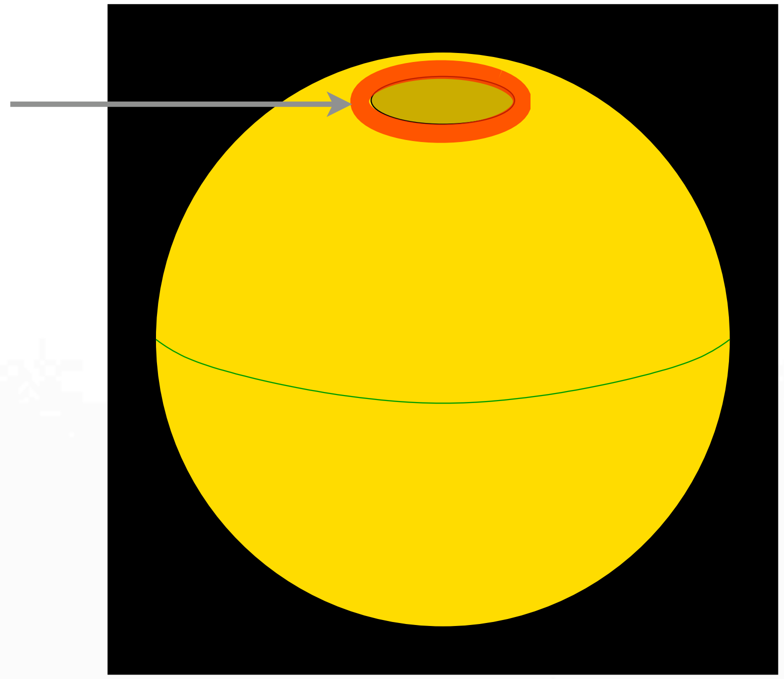
DEMO!!!

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	0	1	0

hot spot for 1-cycle j, where $\Delta_1 j = \lambda j$

hot spot for 2-chain k, where $\Delta_2 k = \lambda k$

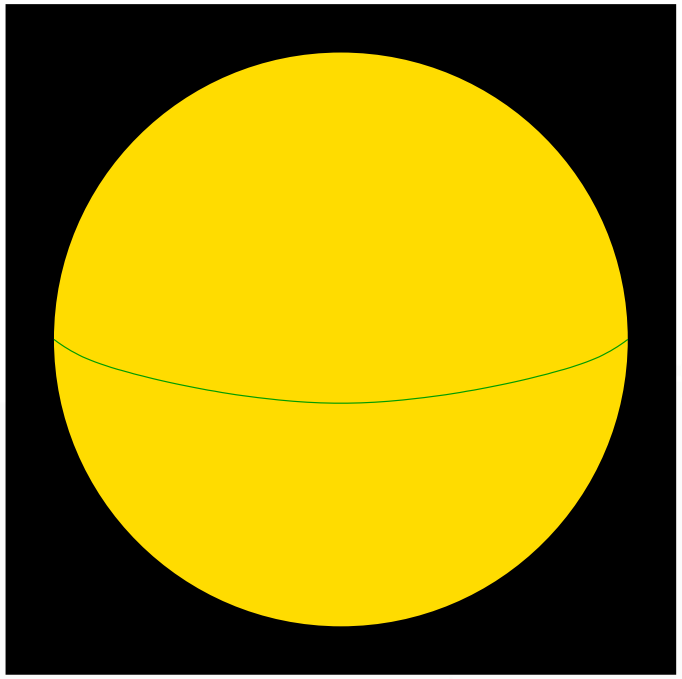
Examples



punctured sphere

b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	0	1	0

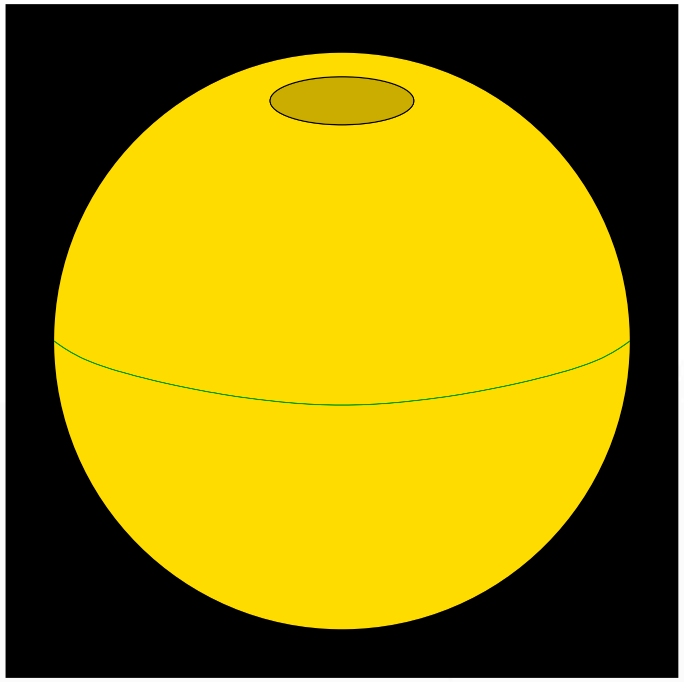
Examples



sphere

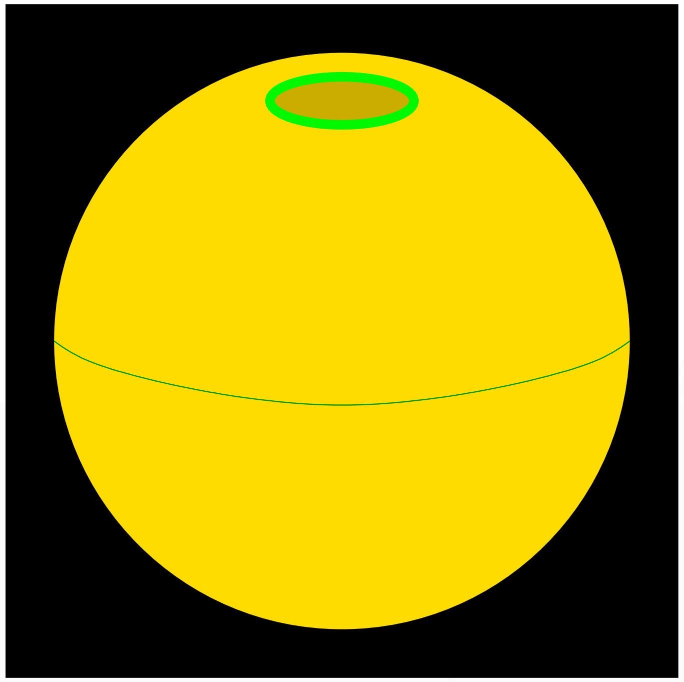
b_0	$b_{0.5}(\epsilon)$	b_1	$b_{1.5}(\epsilon)$	b_2
1	0	0	0	1

What is a (1.5)-D feature?



punctured sphere

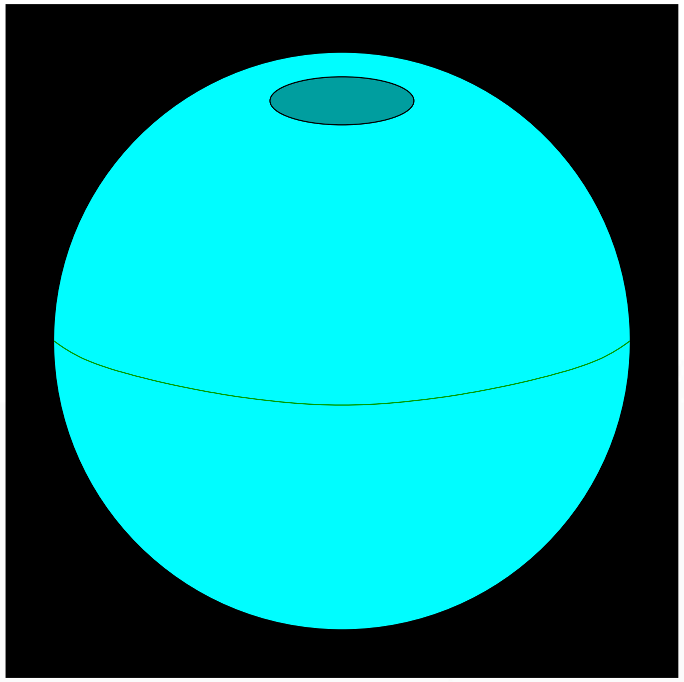
What is a (1.5)-D feature?



punctured sphere

A 1-D cycle which is a boundary (but only just)

What is a (1.5)-D feature?

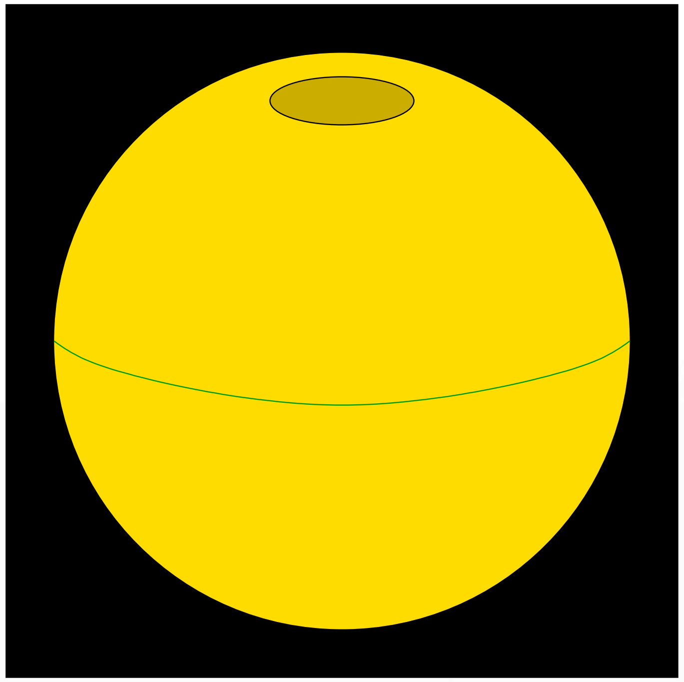


punctured sphere

A 1-D cycle which is a boundary (but only just)

A 2-D chain which is almost (but not quite) closed

What is a (1.5)-D feature?



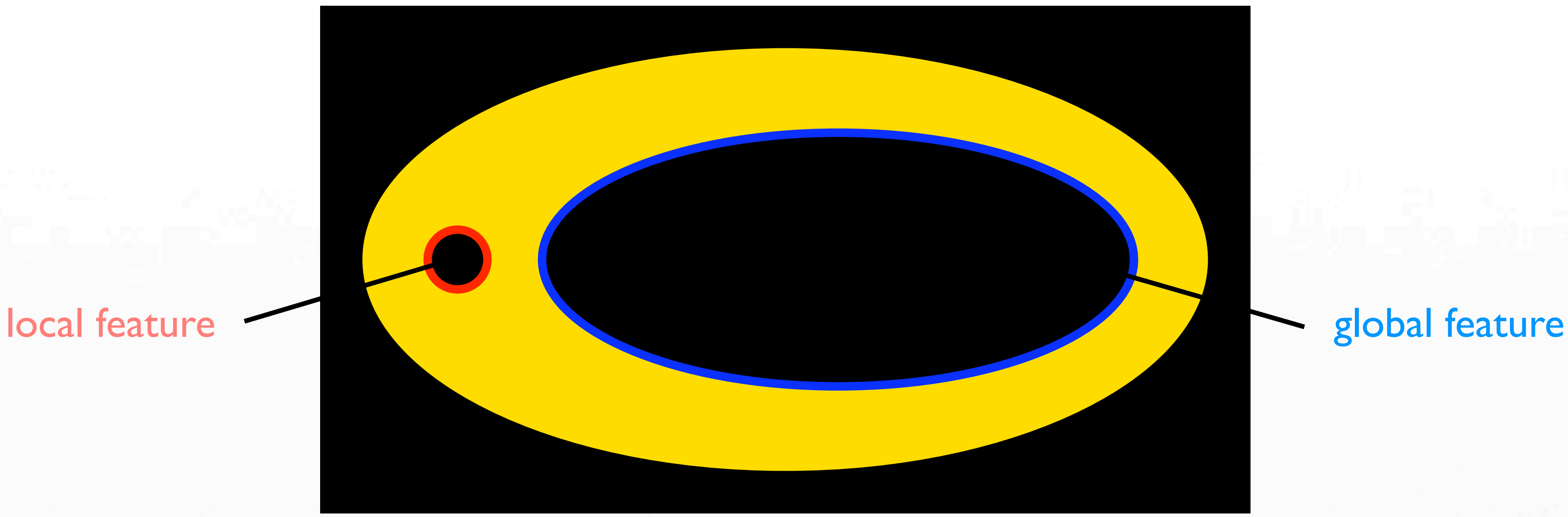
punctured sphere

A 1-D cycle which is a boundary (but only just)

A 2-D chain which is almost (but not quite) closed

Local vs global features

Homological features can be local or global to varying degrees:

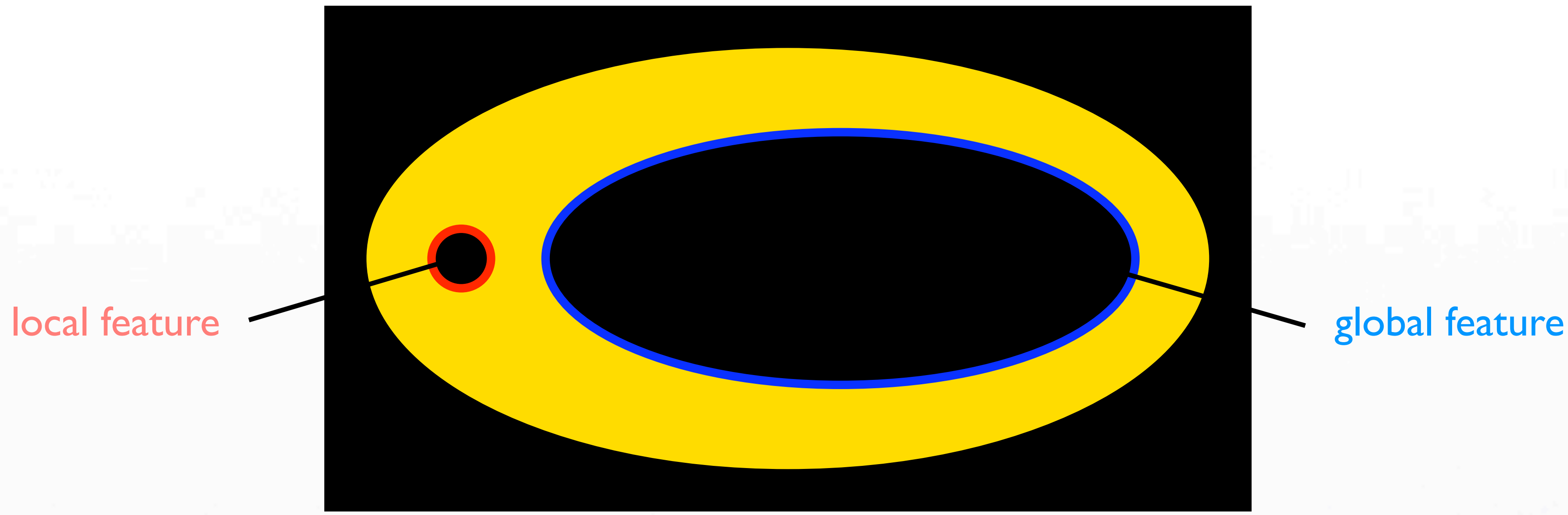


This example has a 2-dimensional space of harmonic 1-forms.

Can we pick out 1-forms representing the two features?

Local vs global features

Homological features can be local or global to varying degrees:



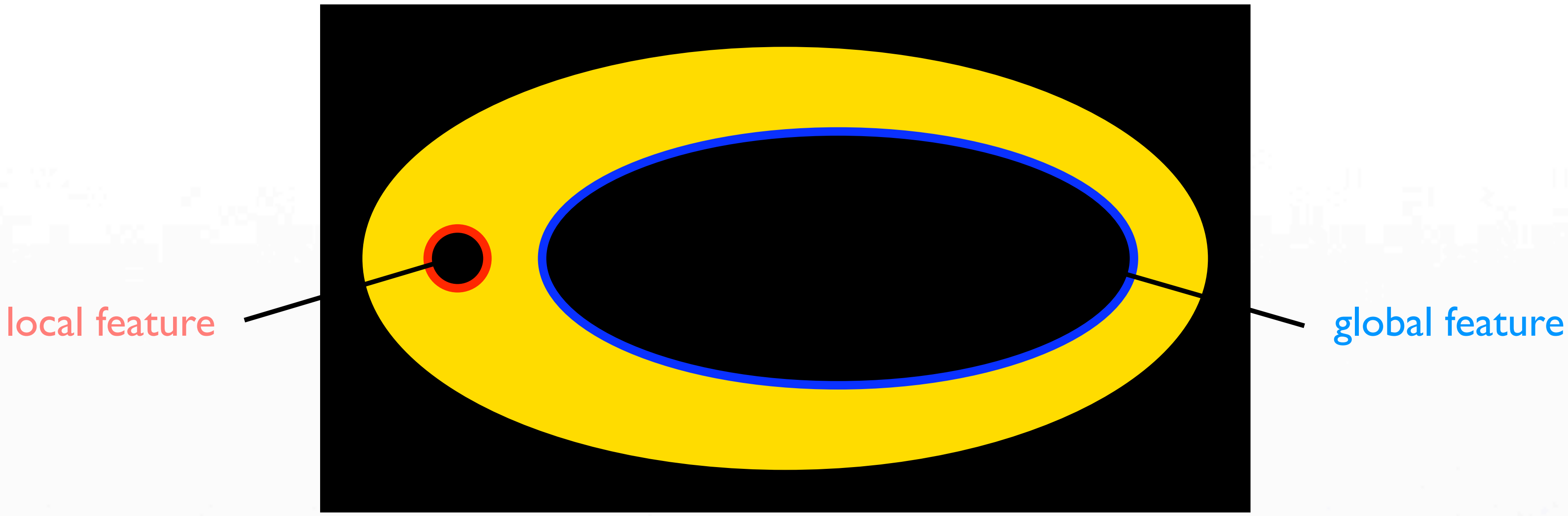
This example has a 2-dimensional space of harmonic 1-forms.

Can we pick out 1-forms representing the two features?

persistent homology can do this very easily

Local vs global features

Homological features can be local or global to varying degrees:



This example has a 2-dimensional space of harmonic 1-forms.

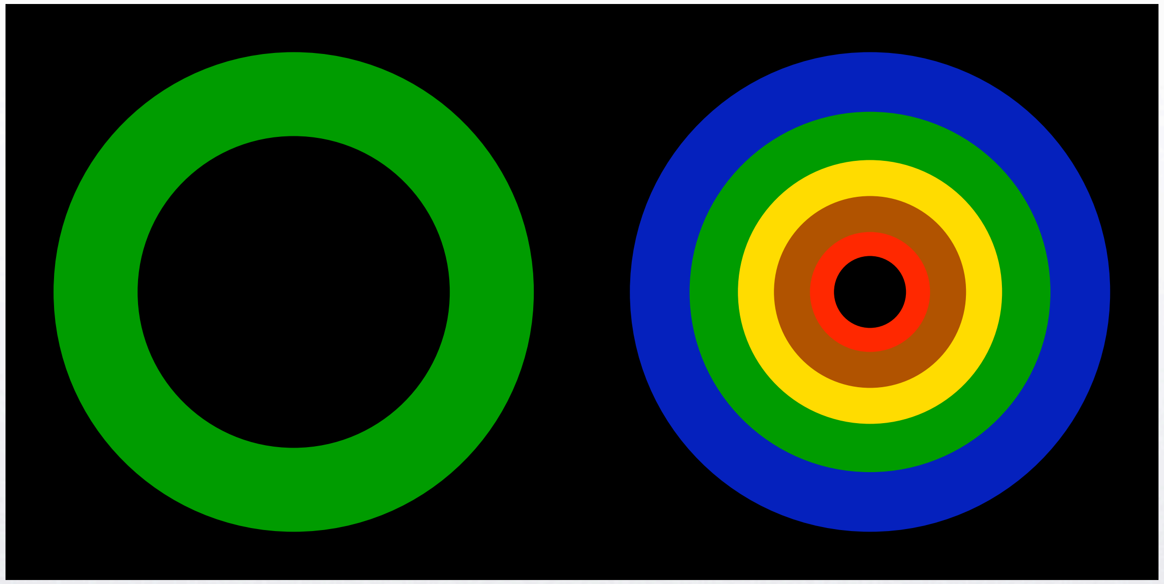
Can we pick out 1-forms representing the two features?

Concentration

- Heuristic arguments suggest that harmonic cycles concentrate energy...
  - ...weakly along global features
  - ...strongly along local features

Concentration

- Heuristic arguments suggest that harmonic cycles concentrate energy...
  - ...weakly along global features
  - ...strongly along local features



Concentration

- Heuristic arguments suggest that harmonic cycles concentrate energy...
  - ...weakly along global features
  - ...strongly along local features

Entropy & L^p comparison

- How to detect whether a cycle is highly concentrated in some region?

- Some measure of entropy is called for
  - high entropy ↔ flat distribution ↔ global feature
  - low entropy ↔ peaked distribution ↔ local feature

- Simple estimate: compare L^1 and L^2 norms
  - $E[f] := ||f||_1 / ||f||_2$
  - E[f] large ↔ global feature
  - E[f] small ↔ local feature

Entropy & L^p comparison

- How to detect whether a cycle is highly concentrated in some region?

- Some measure of entropy is called for
  - high entropy ↔ flat distribution ↔ global feature
  - low entropy ↔ peaked distribution ↔ local feature

- Simple estimate: compare L^1 and L^2 norms
  - $E[f] := ||f||_1 / ||f||_2$
  - E[f] large ↔ global feature
  - E[f] small ↔ local feature

DEMO!!!

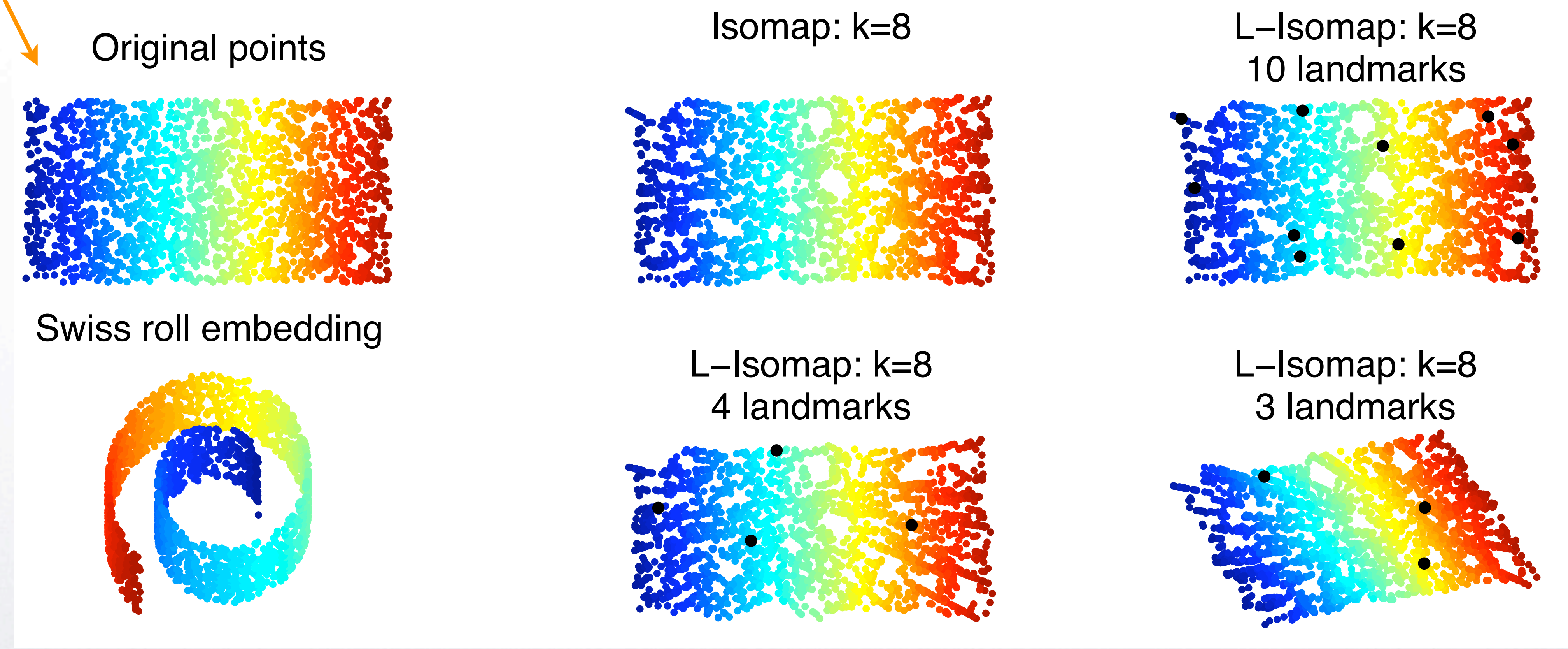
Cohomology and S^1-functions

Joint work with Mikael Vejdemo-Johansson

Isomap (etc)

Nonlinear Dimensionality Reduction

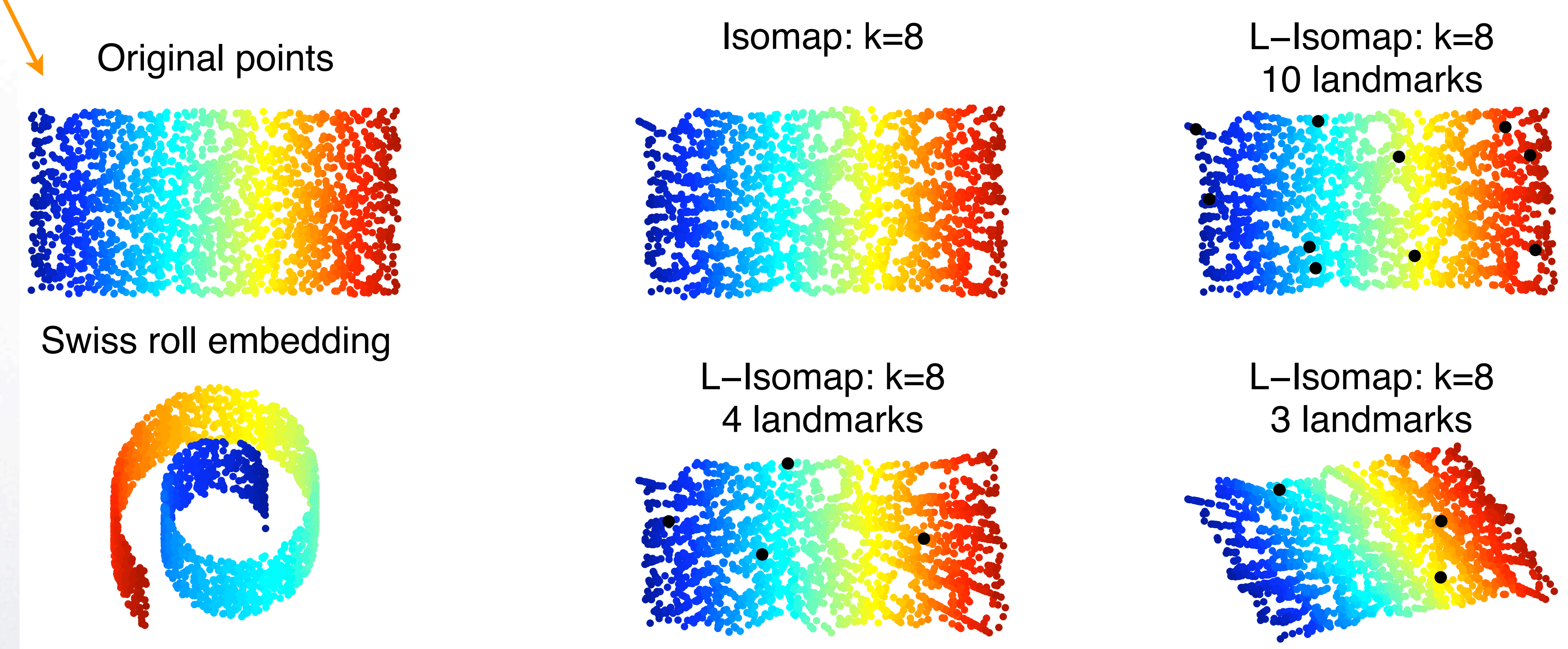
unknown: linear parameter space



Isomap (etc)

Nonlinear Dimensionality Reduction

unknown: linear parameter space

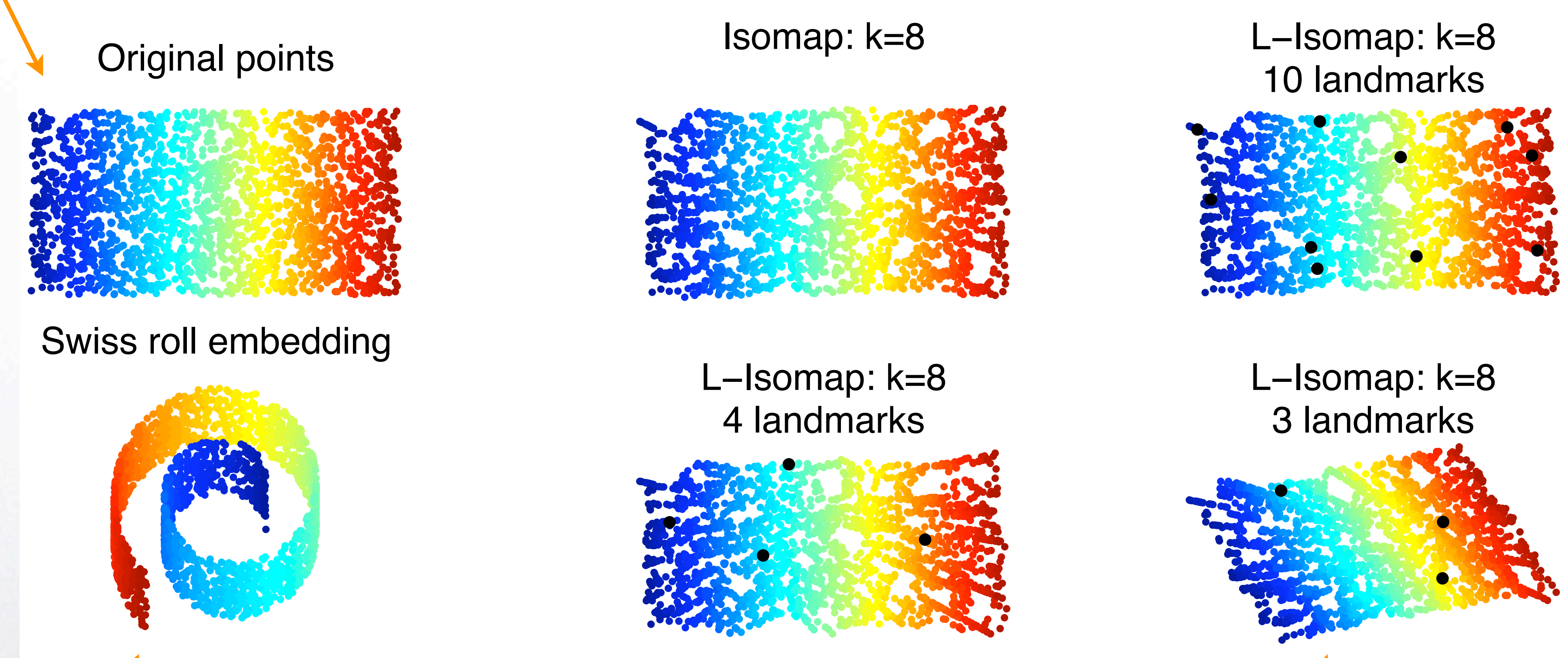


input: nonlinear observed data

Isomap (etc)

Nonlinear Dimensionality Reduction

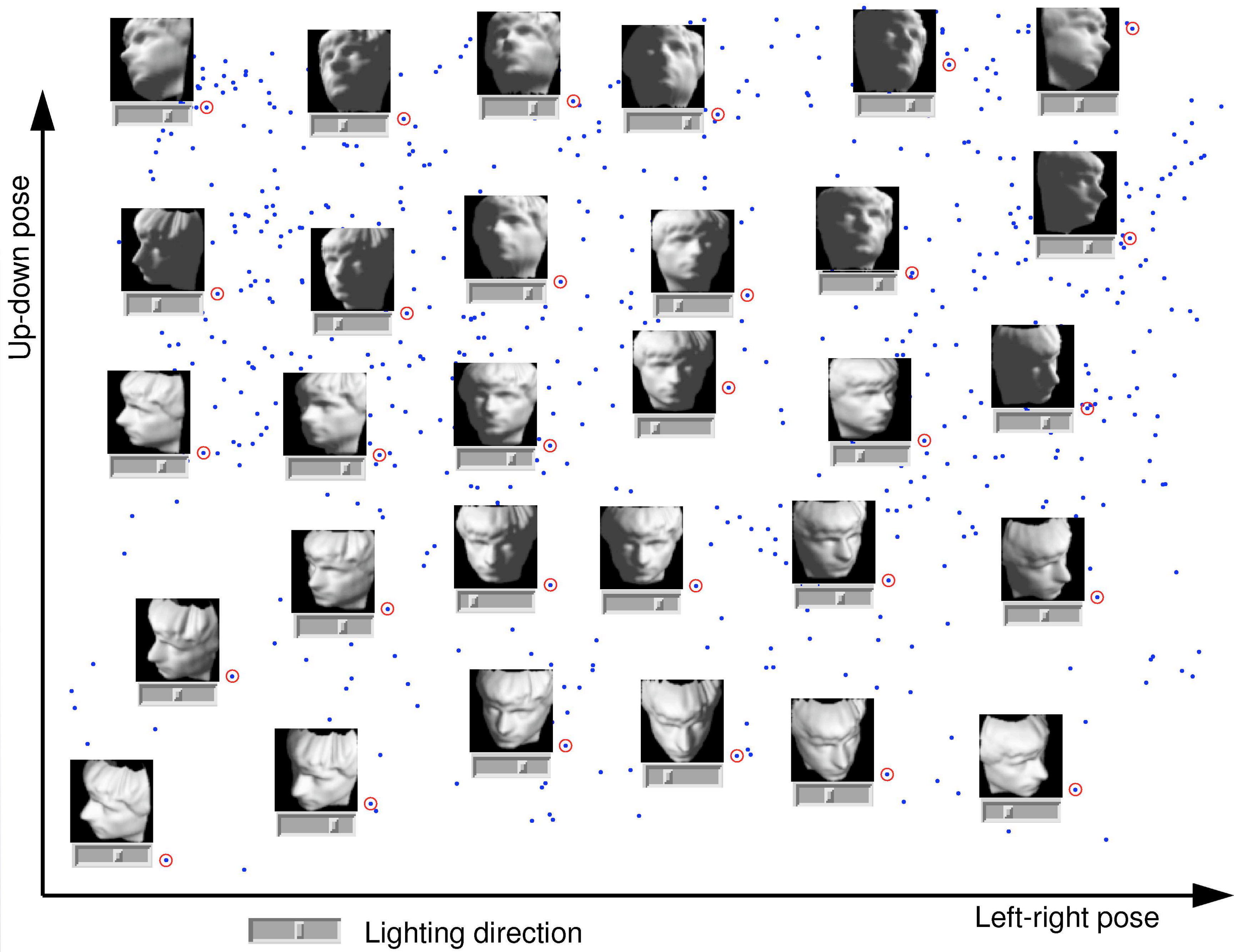
unknown: linear parameter space



input: nonlinear observed data

output: low-dimensional coordinate embedding

Example: face images



NLDR techniques

- Beginning December 2000:
  - Isomap (Tenenbaum, dS, Langford)
  - LLE (Roweis, Saul)
  - Laplacian Eigenmaps (Belkin, Niyogi)
  - Hessian Eigenmaps (Donoho, Grimes)
  - ...
- Goal: find useful real-valued coordinate functions on data
  - Most effective when data lie on the image of a convex region
  - Nontrivial topology typically causes problems

NLDR techniques

actually, 1997

- Beginning December 2000:
  - Isomap (Tenenbaum, dS, Langford)
  - LLE (Roweis, Saul)
  - Laplacian Eigenmaps (Belkin, Niyogi)
  - Hessian Eigenmaps (Donoho, Grimes)
  - ...

- Goal: find useful real-valued coordinate functions on data
  - Most effective when data lie on the image of a convex region
  - Nontrivial topology typically causes problems

NLDR techniques

actually, 1997

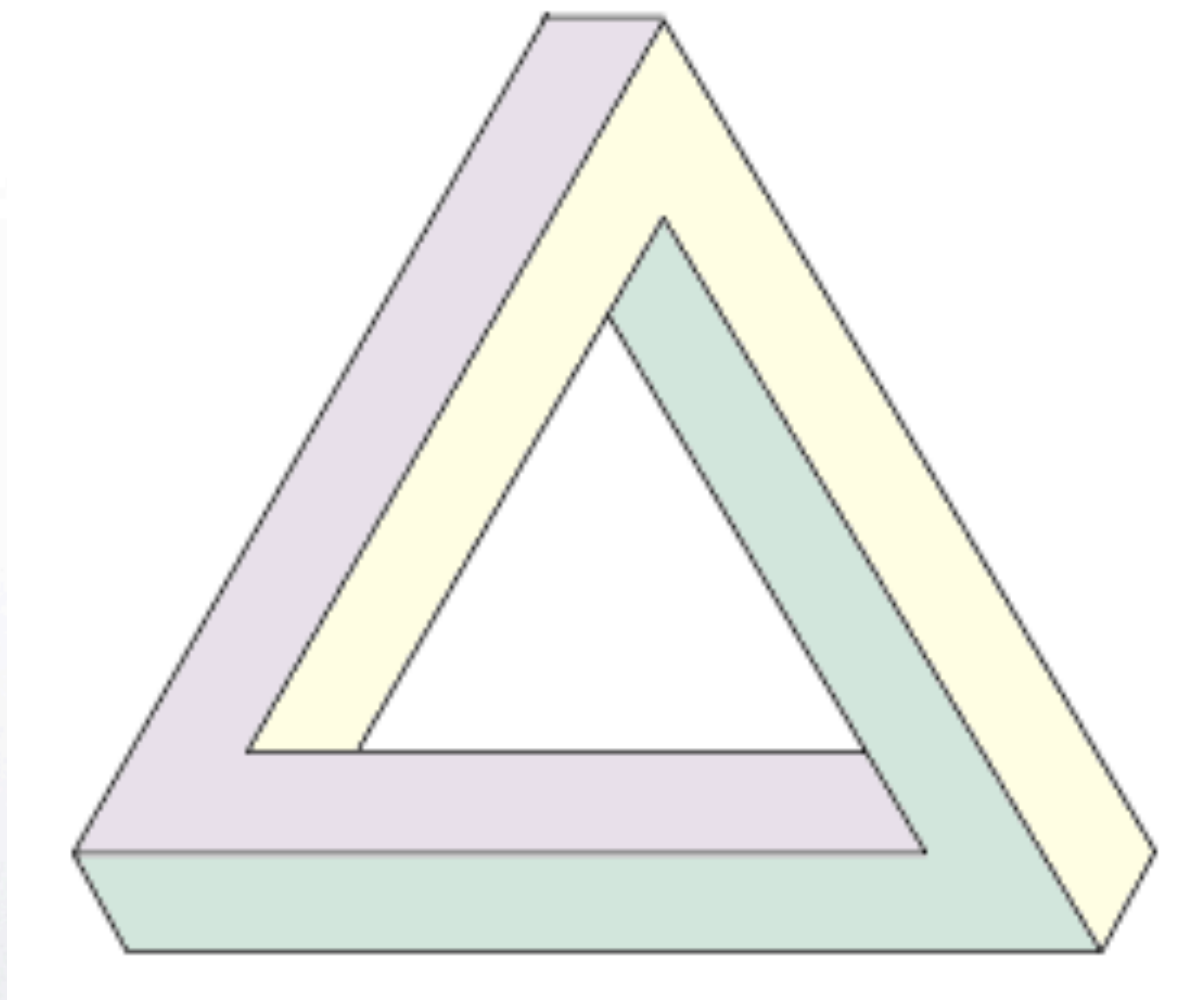
- Beginning December 2000:
  - Isomap (Tenenbaum, dS, Langford)
  - LLE (Roweis, Saul)
  - Laplacian Eigenmaps (Belkin, Niyogi)
  - Hessian Eigenmaps (Donoho, Grimes)
  - ...

- Goal: find useful real-valued coordinate functions on data
  - Most effective when data lie on the image of a convex region
  - Nontrivial topology typically causes problems

What about circle-valued coordinates? $\vartheta_i : X \rightarrow S^1$

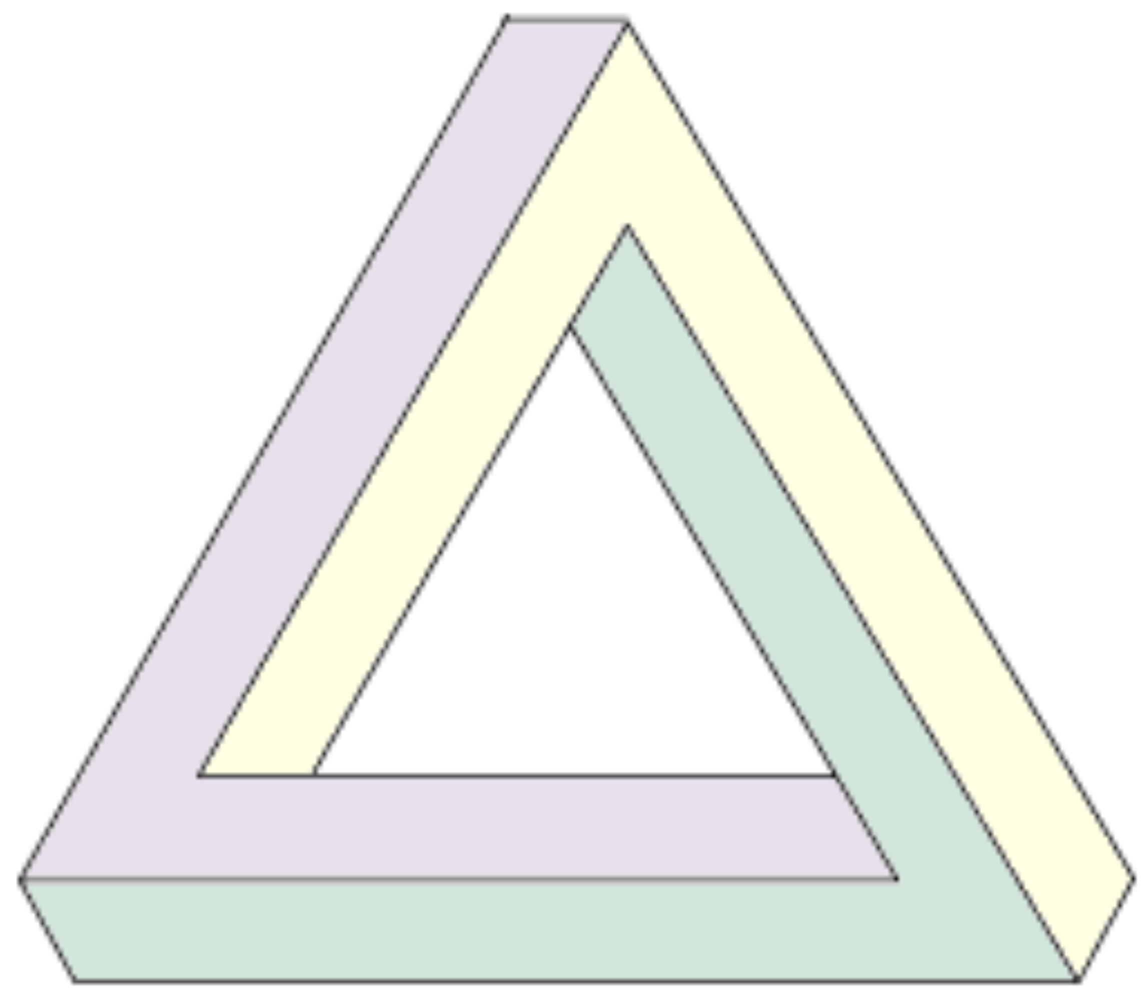
Vin de Silva
http://pages.pomona.edu/~vds04747/

An idea of Penrose...



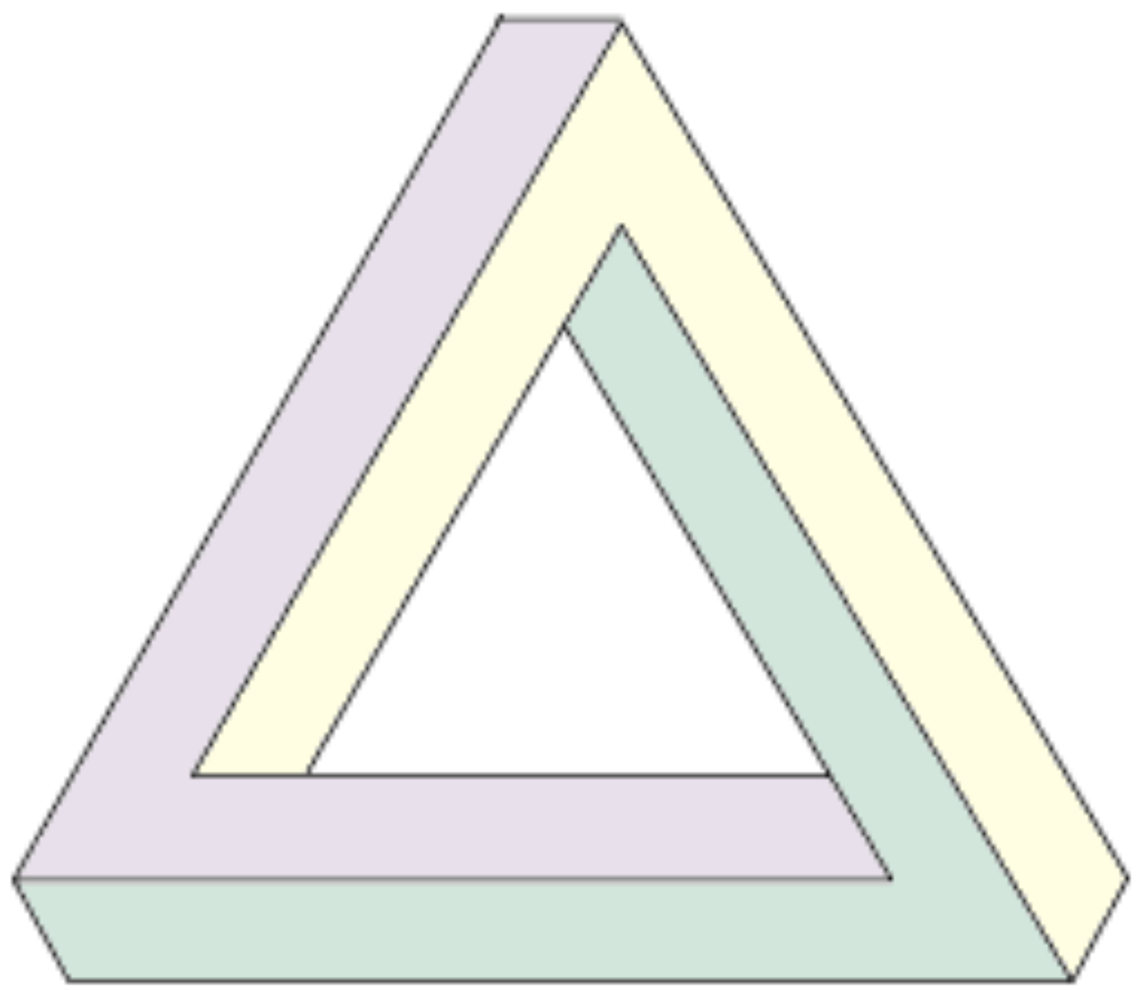
An idea of Penrose...

- What is the depth function f(x)?
- f(x)-f(y) is locally consistently defined.
- There is no global f(x).



An idea of Penrose...

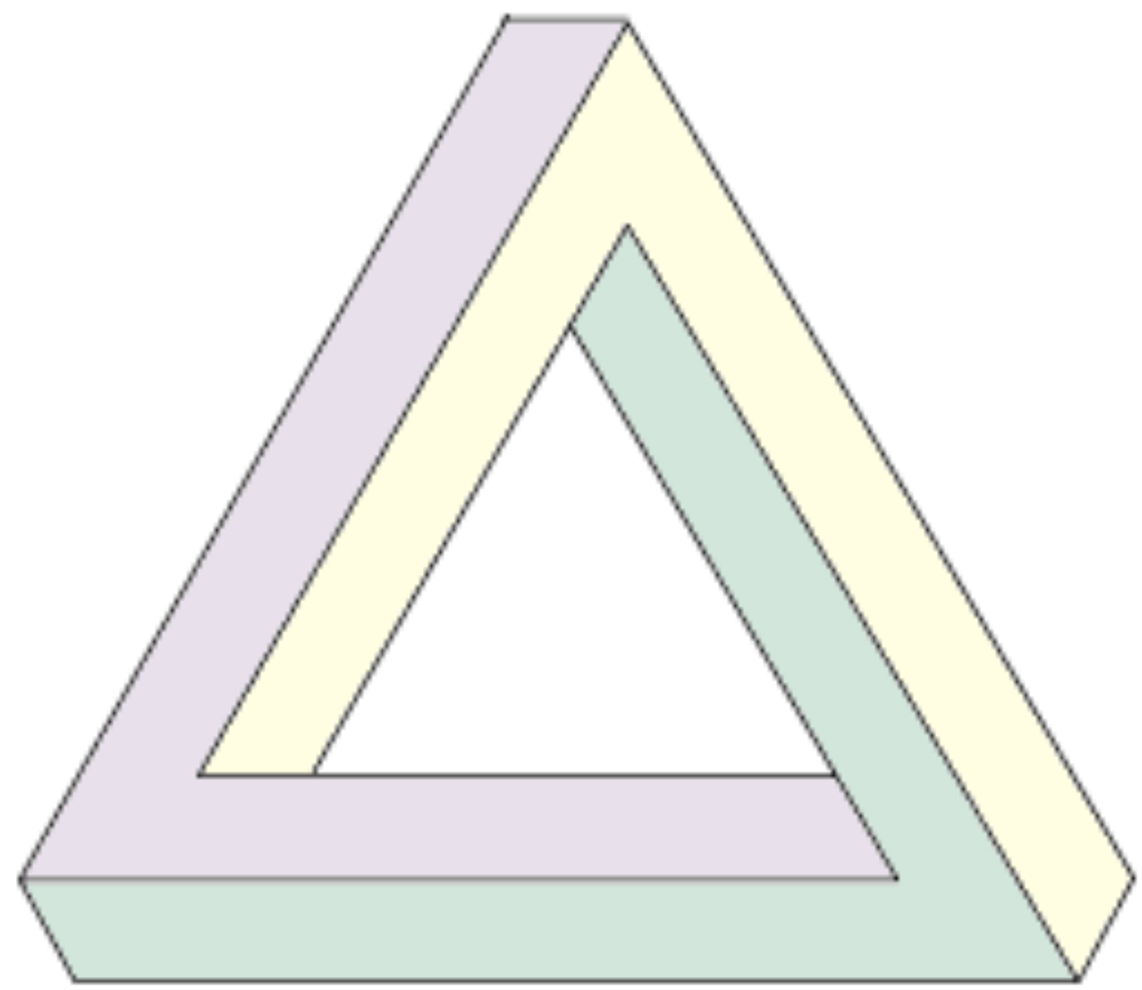
- What is the depth function f(x)?
- f(x)-f(y) is locally consistently defined.
- There is no global f(x).



cohomology class in $H^1(X)$

An idea of Penrose...

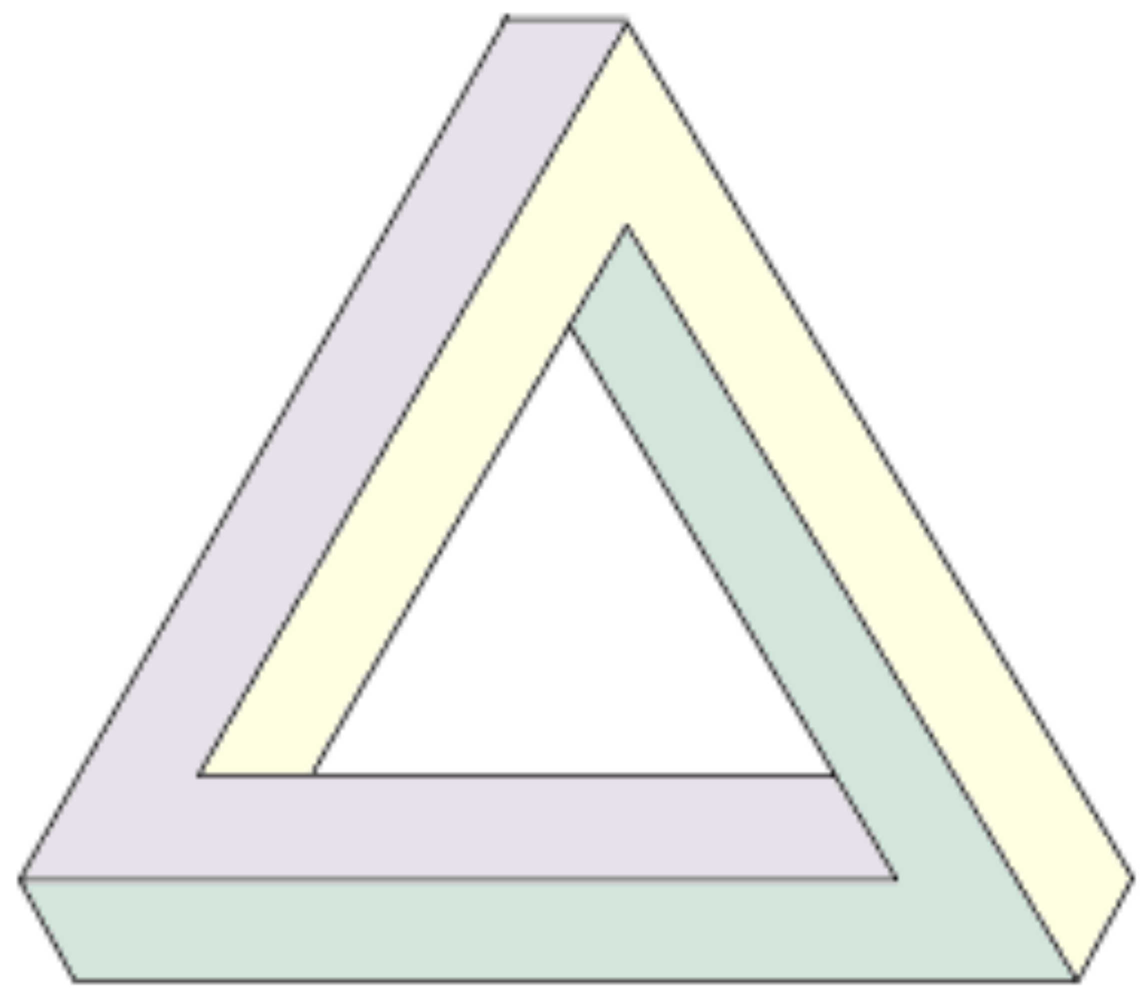
- What is the depth function f(x)?
- f(x)-f(y) is locally consistently defined.
- There is no global f(x).



nonzero cohomology class in $H^1(X)$

An idea of Penrose...

- What is the depth function f(x)?
- f(x)-f(y) is locally consistently defined.
- There is no global f(x).



nonzero cohomology class in $H^1(X)$

circle-valued depth function

Vin de Silva
http://pages.pomona.edu/~vds04747/

Homotopy theory

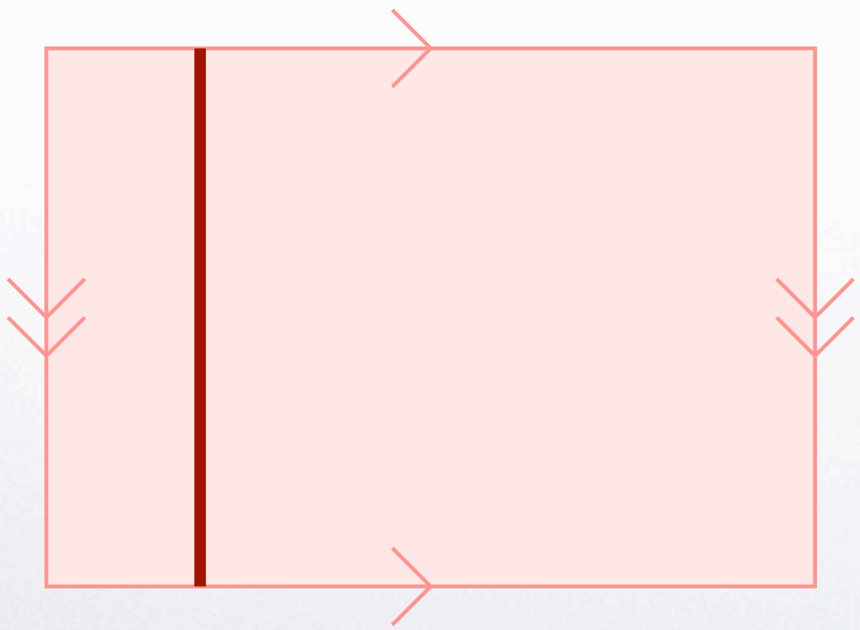
$$[X, S^1] = H^1(X; Z)$$

Homotopy classes of maps $X \to S^1$

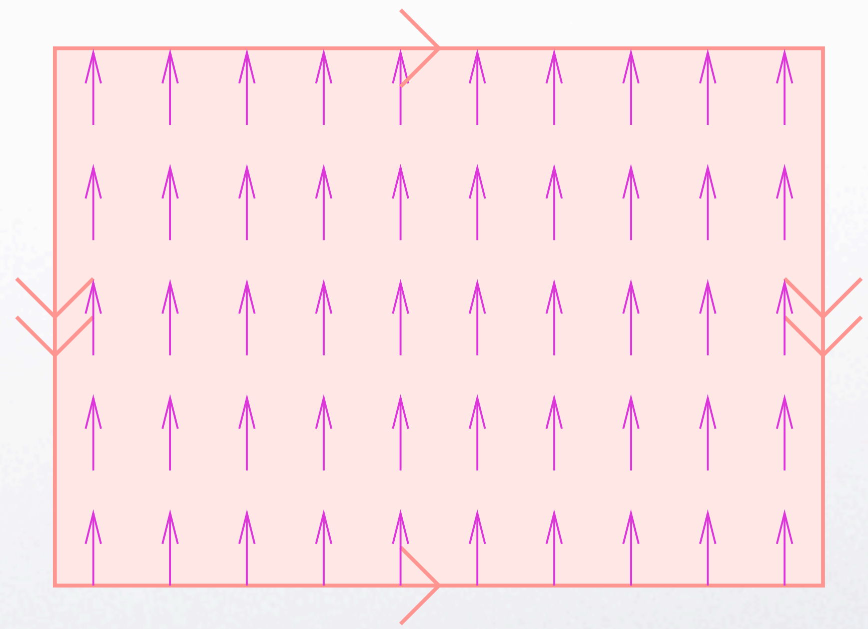
Vin de Silva
http://pages.pomona.edu/~vds04747/

Why Cohomology?

- homology, homotopy: maps into X
- cohomology, cohomotopy: maps from X

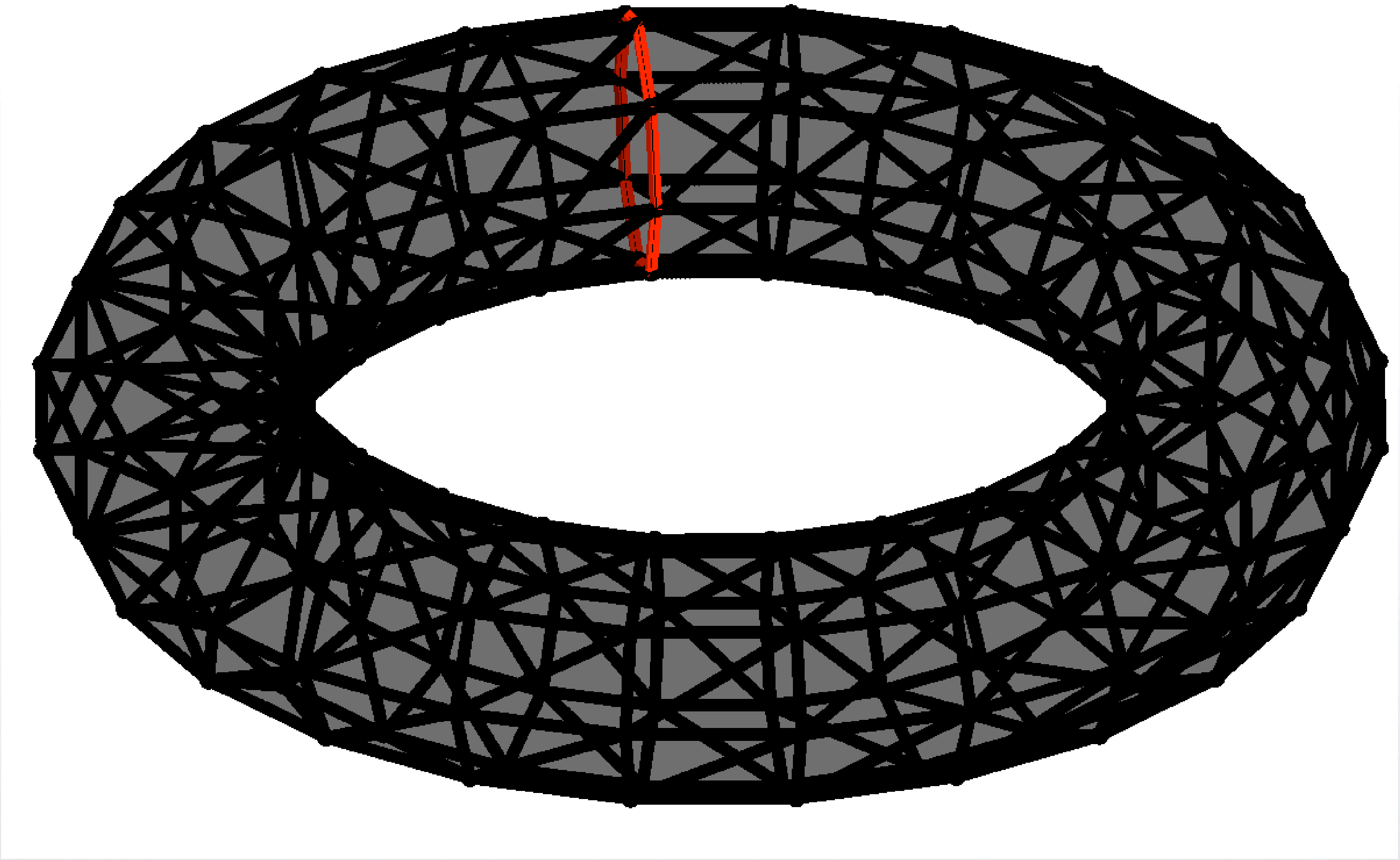


1-cycle

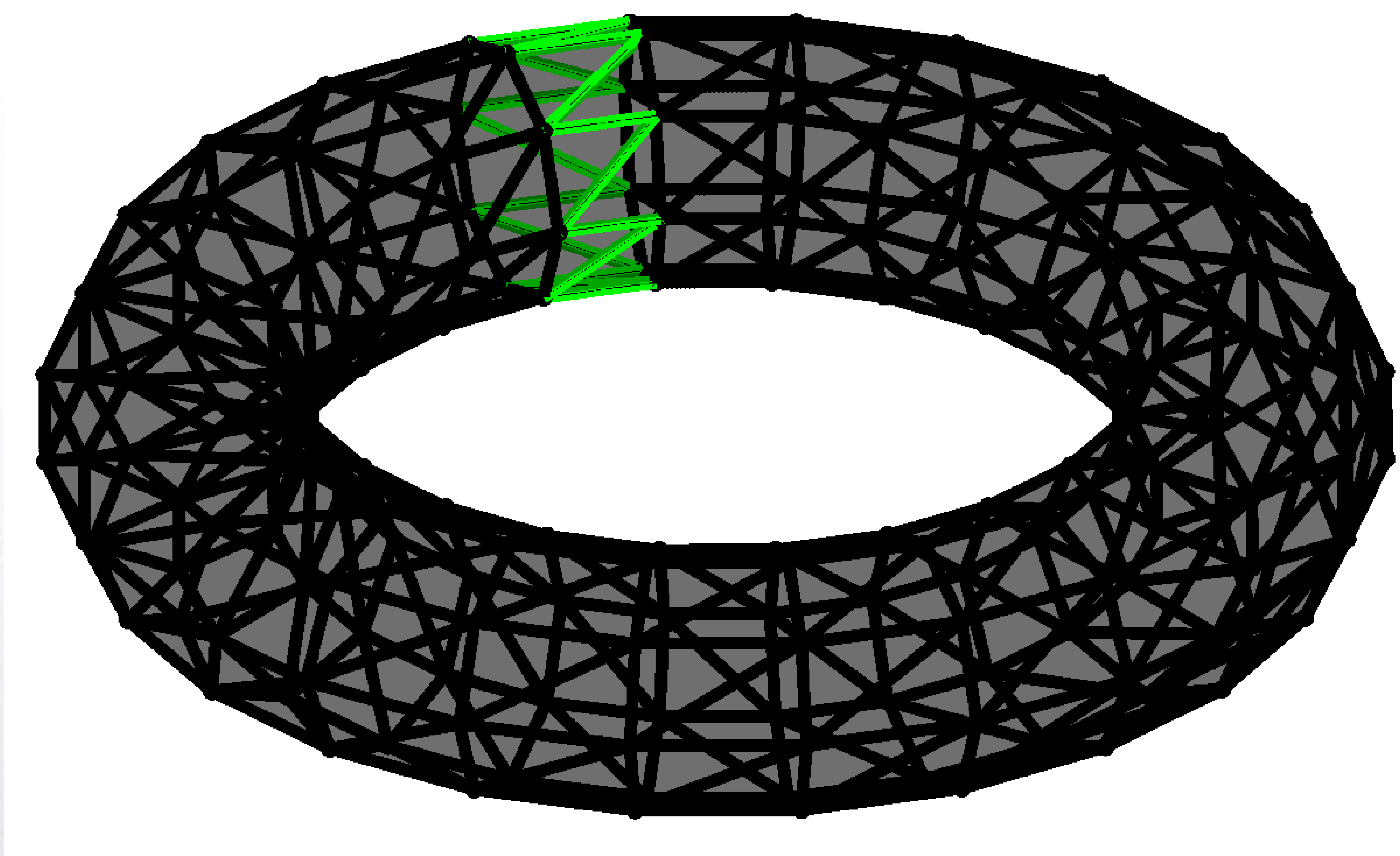


de Rham 1-cocycle

Homology

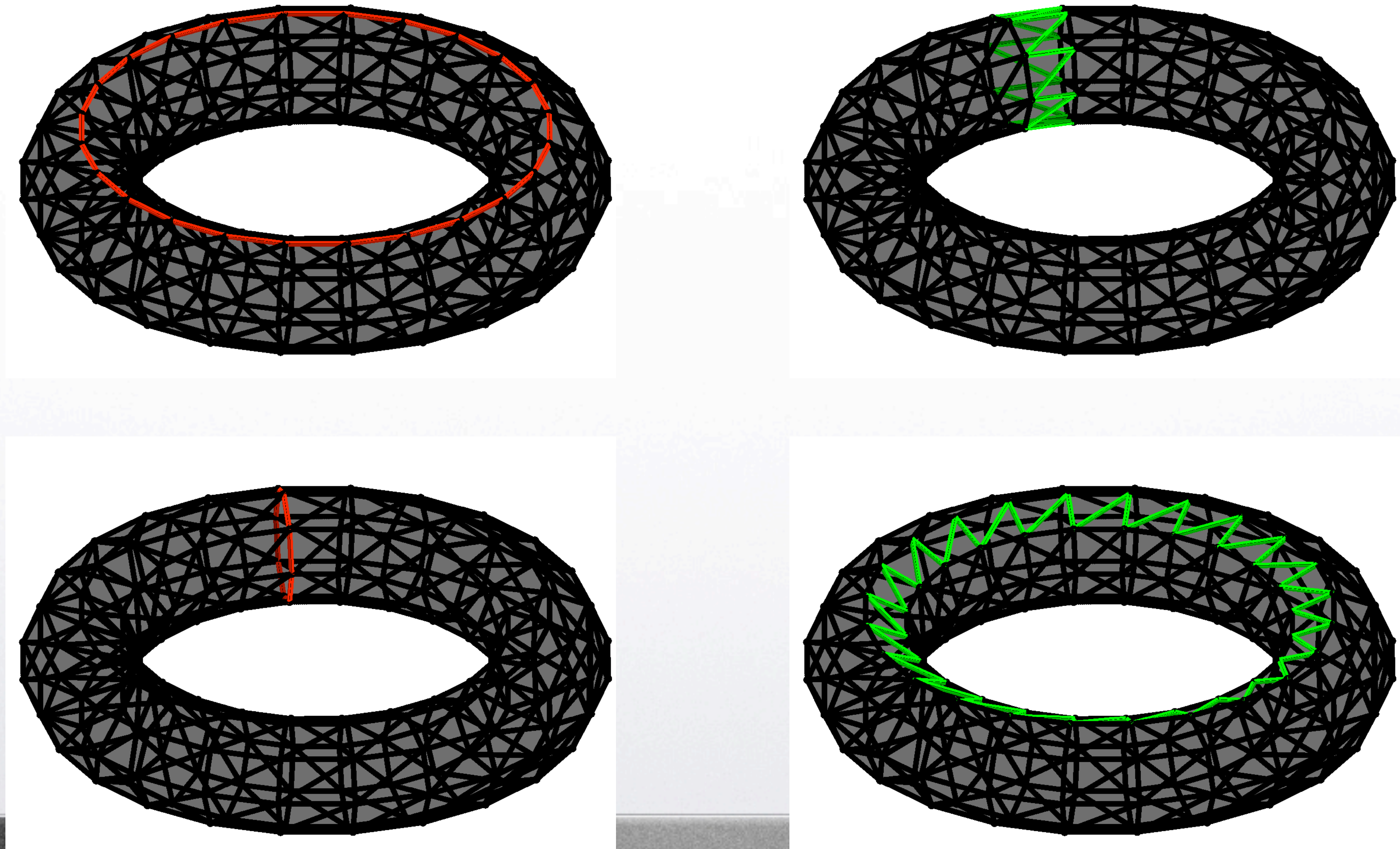


Cohomology

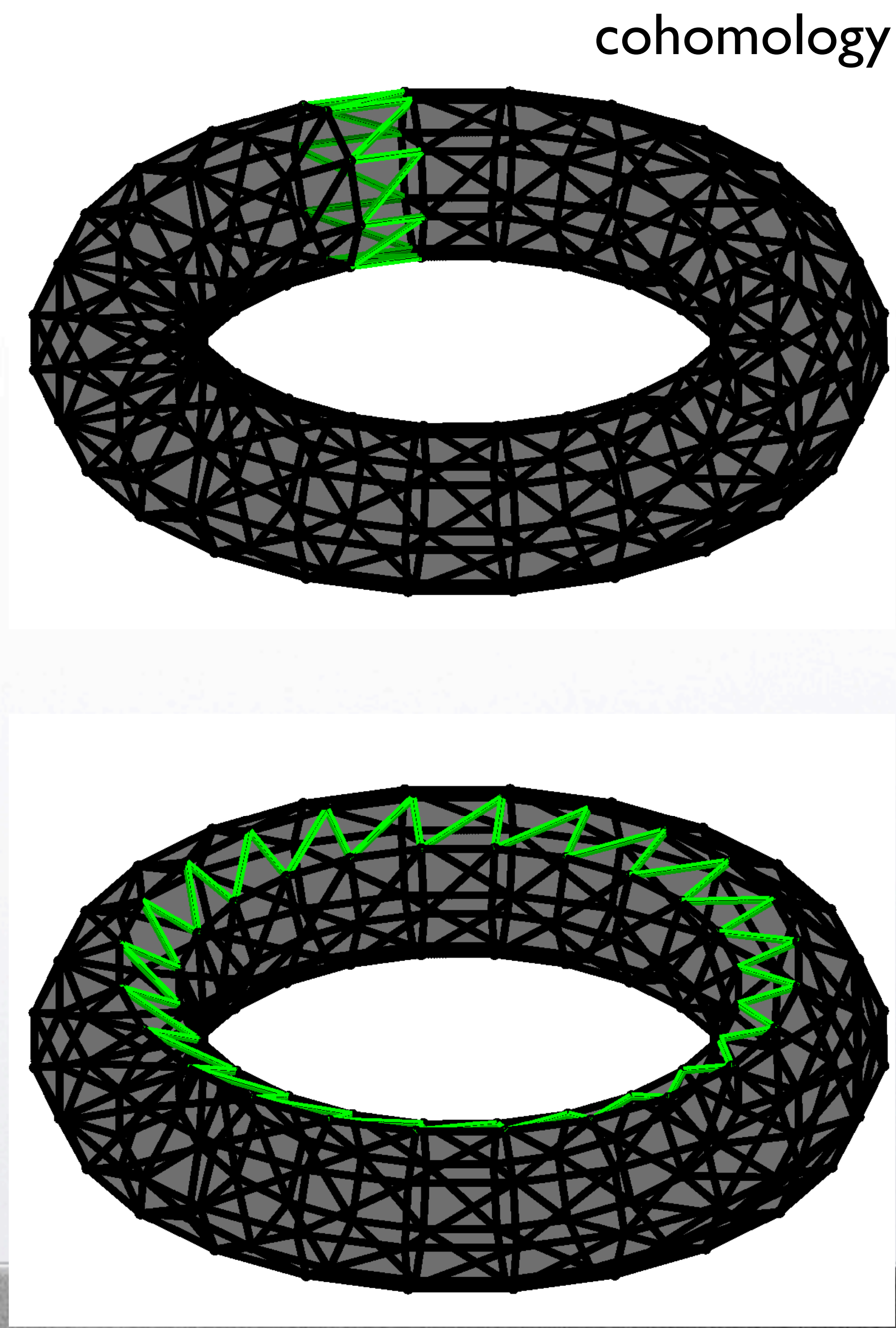


Dual bases

homology | cohomology

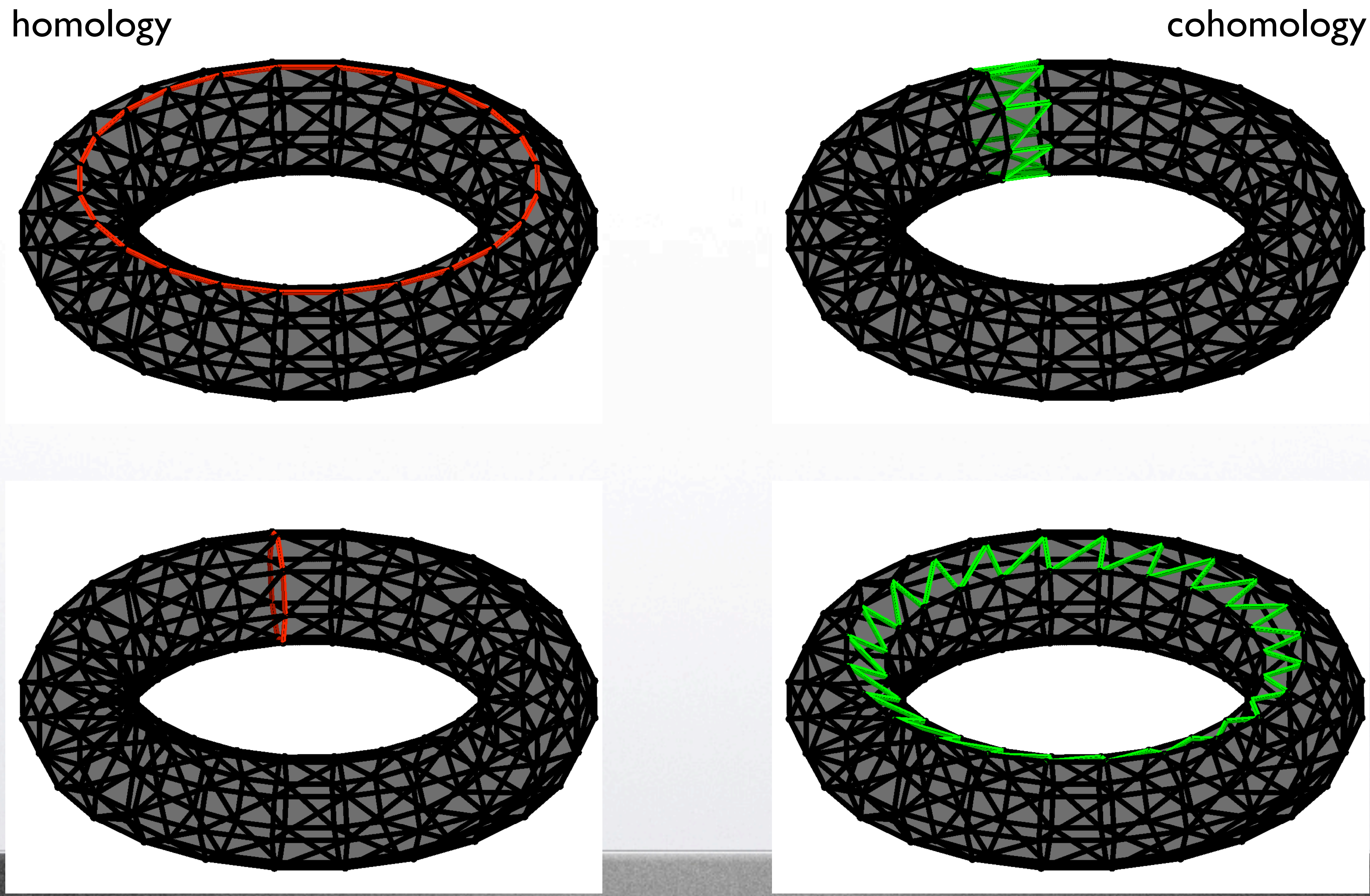


Circle maps



- Integer cocycles give rise to circle maps...
- ...but these are abrupt and unsmooth.

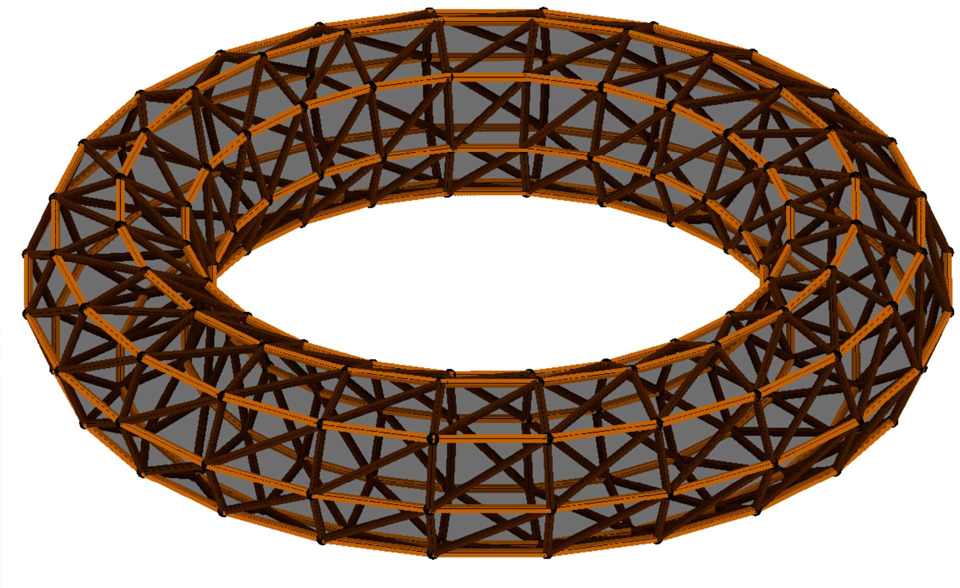
Harmonic smoothing

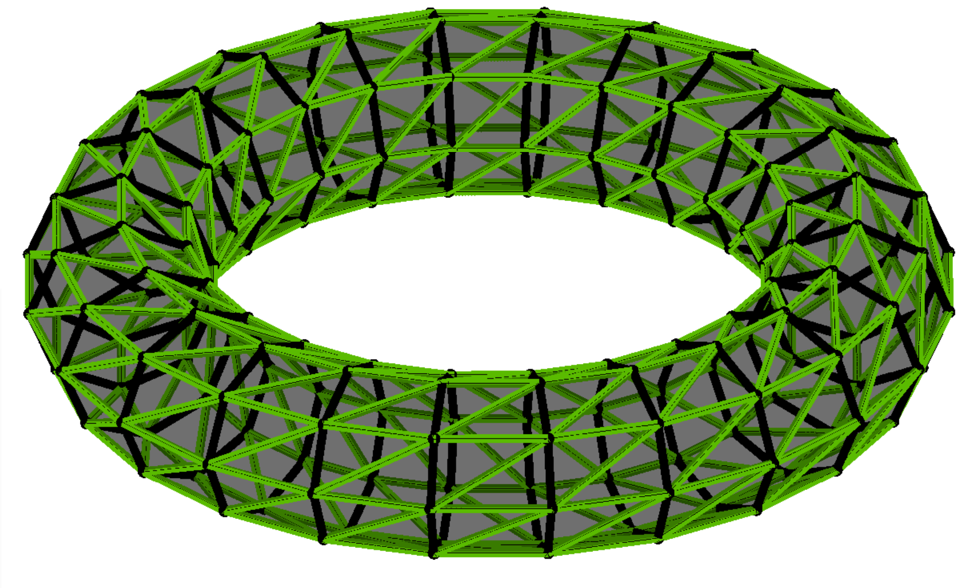


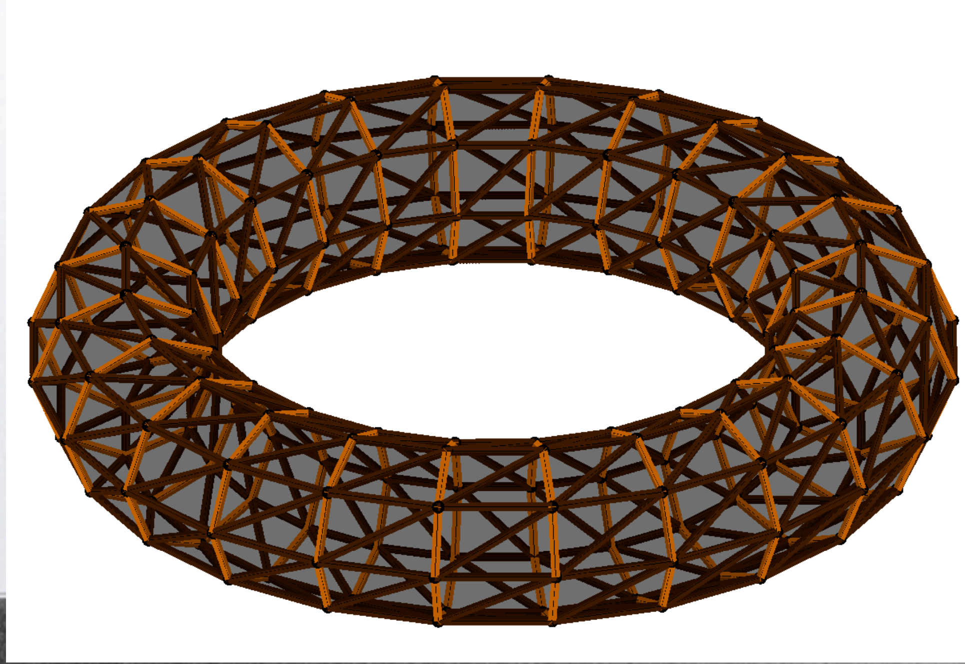
Harmonic smoothing

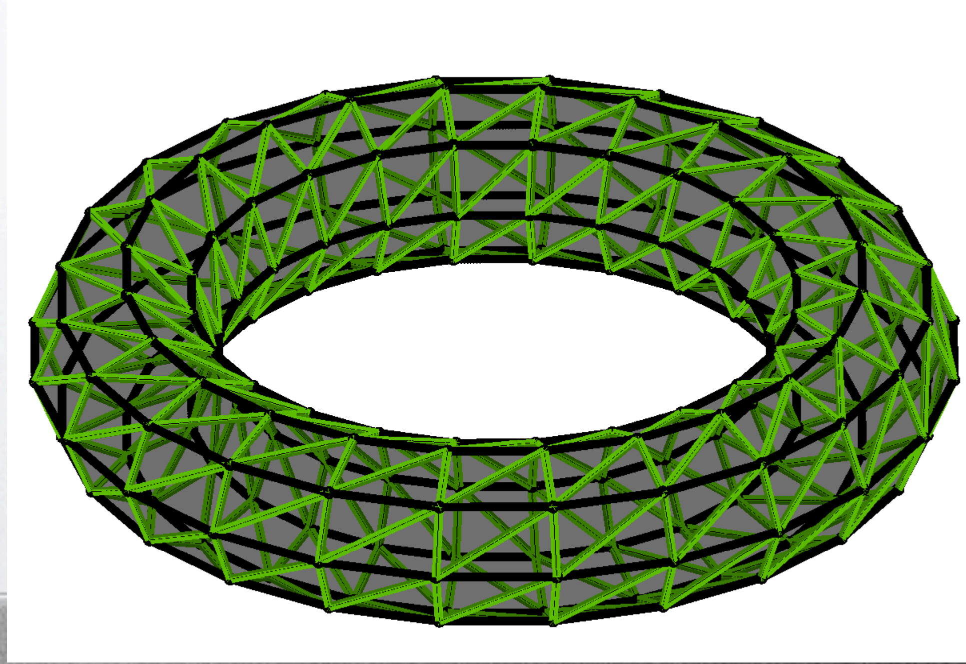
homology

cohomology





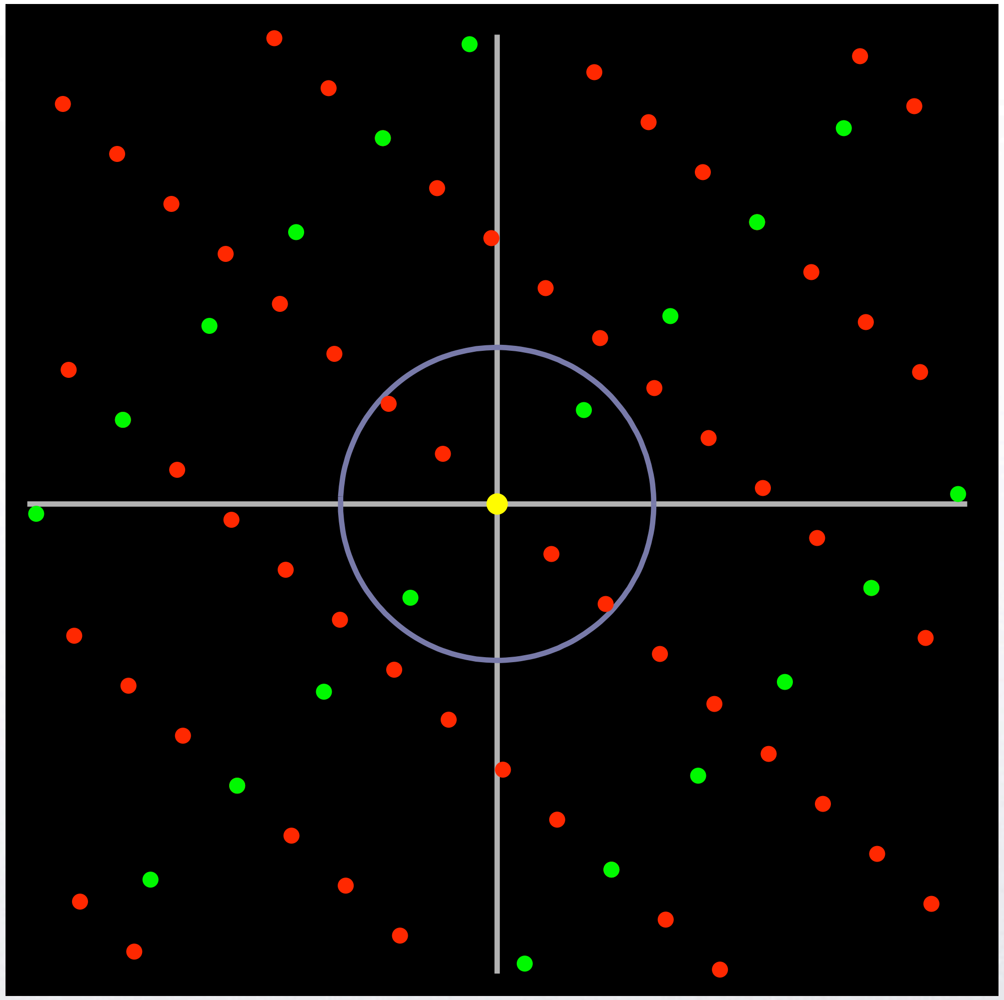




Dual lattices

- After smoothing, integer cocycles become real cocycles...
- ...but they still produce circle maps.
- Seek harmonic forms in integer cohomology lattice.

harmonic 1-forms

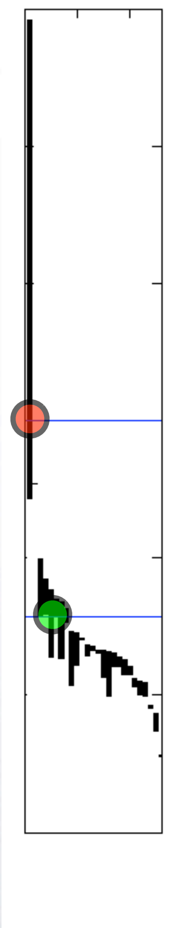


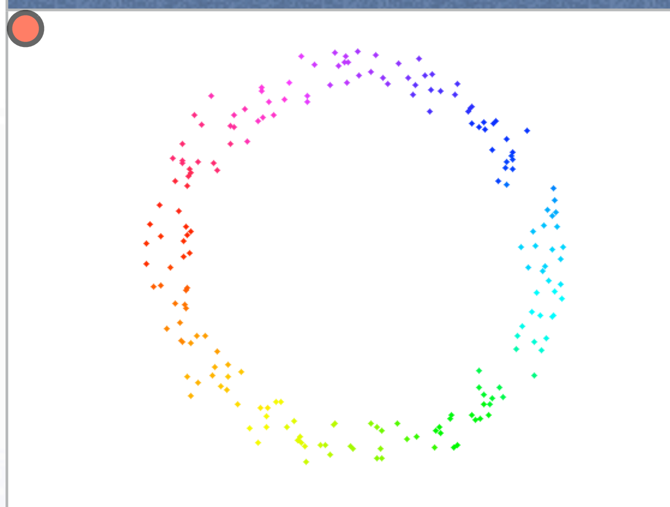
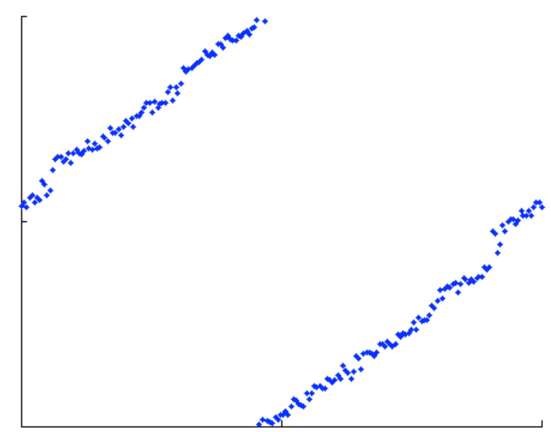
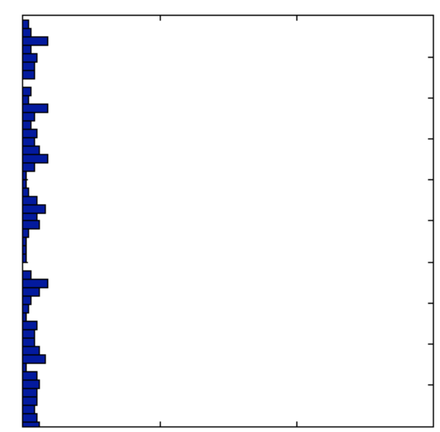
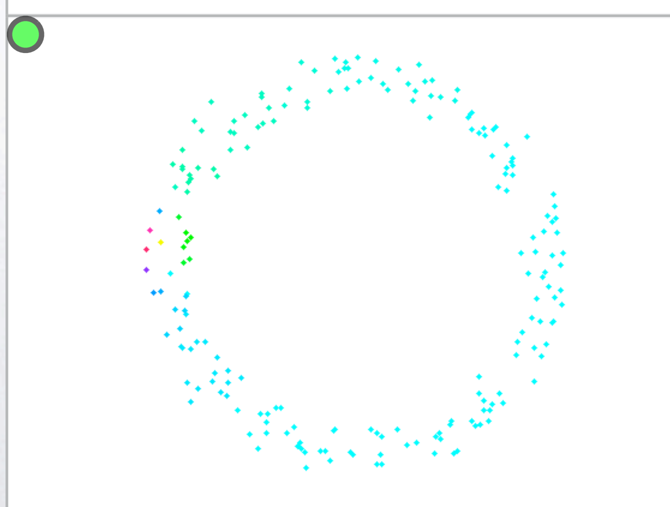
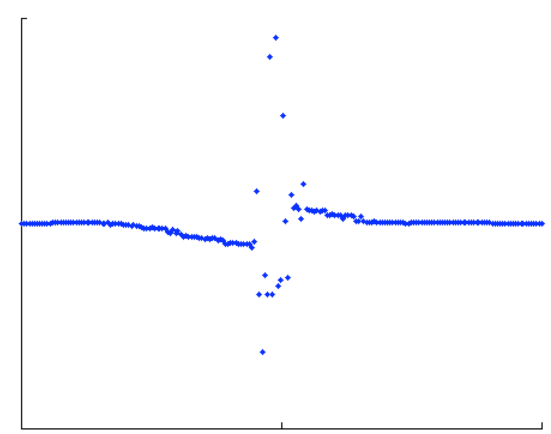
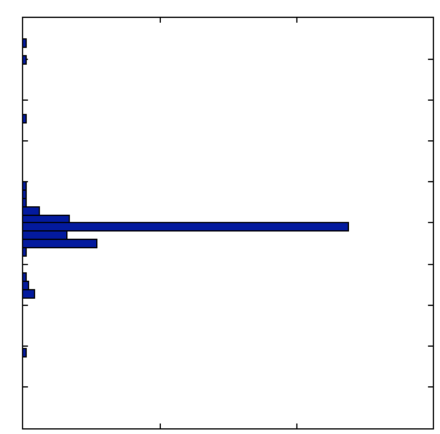
integer homology and cohomology lattices

Strategy

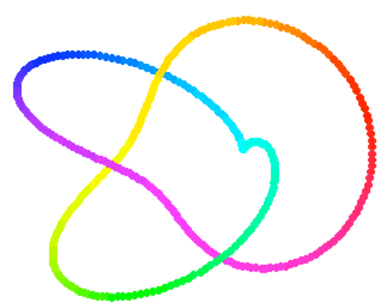
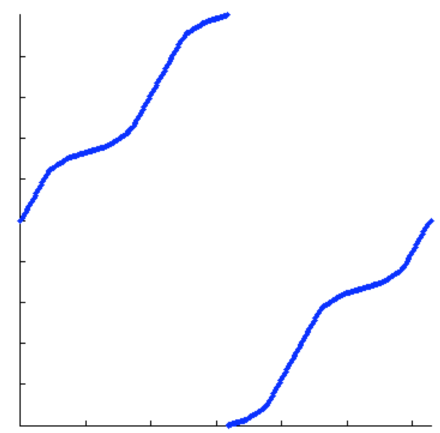
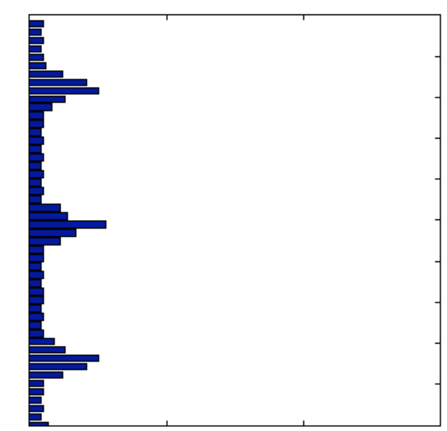
- Filtered complex — $\mathbb{X} = \{X^\epsilon\}_{\epsilon \geq 0}$
- Persistent cohomology (mod p) — $\mathrm{pH}^1(\mathbb{X}; \mathbb{F}_p)$
- Select significant cocycle — $[\alpha_p] \in H^1(X^\epsilon; \mathbb{F}_p)$
- Lift to integer coefficients — $[\alpha] \in H^1(X^\epsilon; \mathbb{Z})$
- Smooth — $\bar{\alpha} \in \mathcal{H}^1(X^\epsilon) \subseteq C^1(X^\epsilon; \mathbb{R})$
- Integrate — $\theta : X^\epsilon \to S^1$

Noisy circle

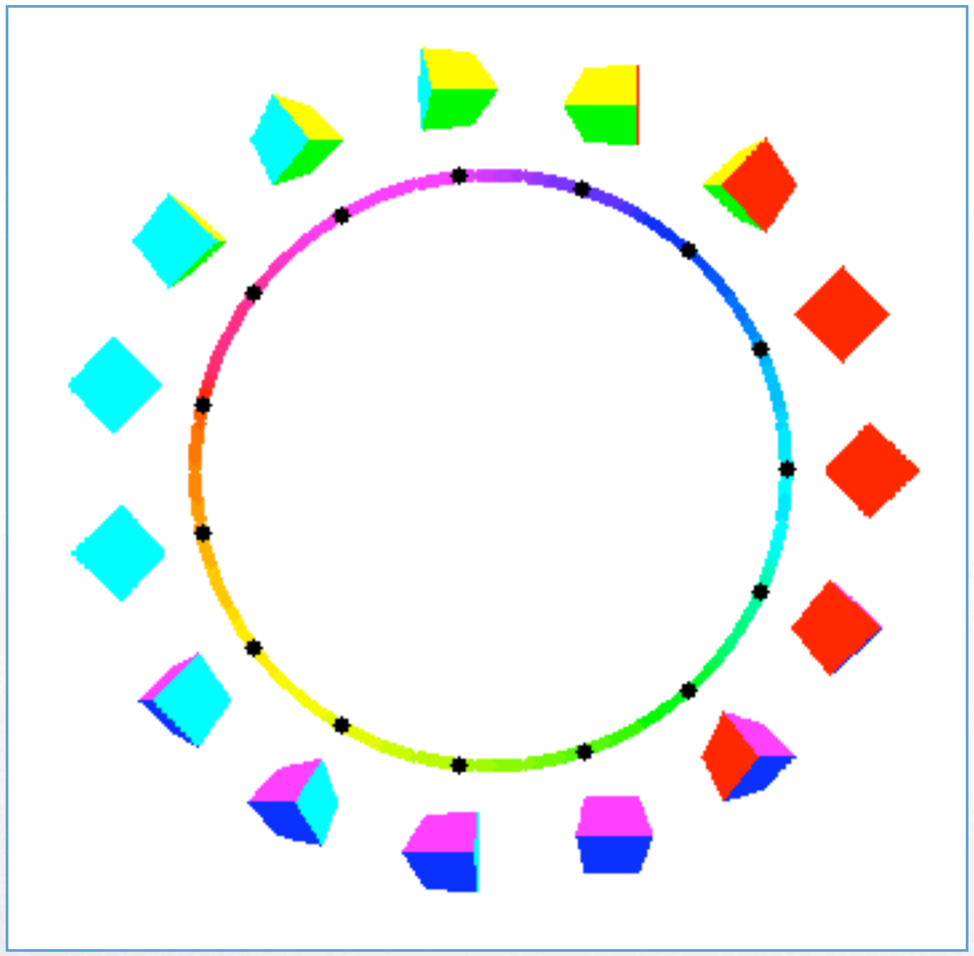


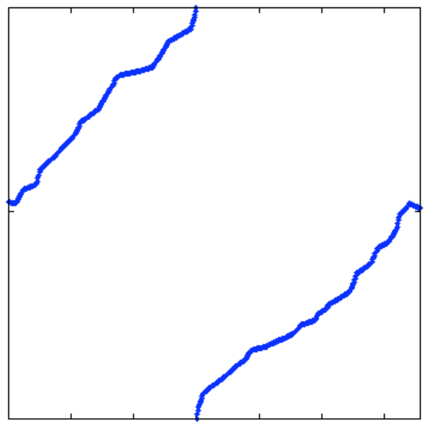
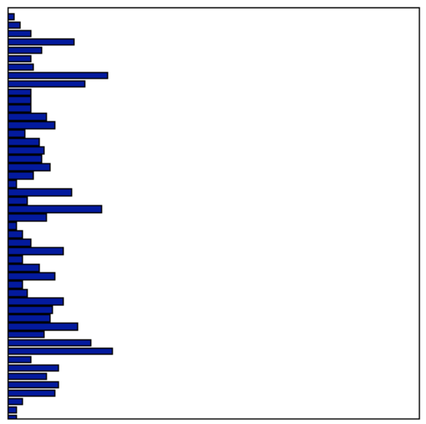
inferred	inferred vs original	histogram
		
		

Trefoil knot

inferred	inferred vs original	histogram
		

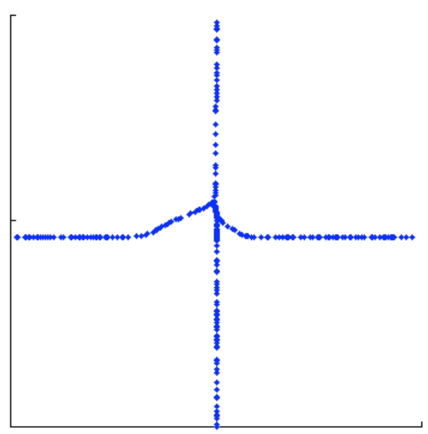
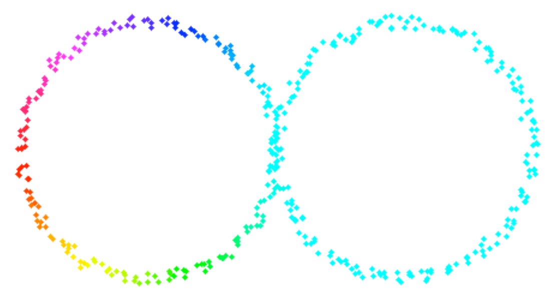
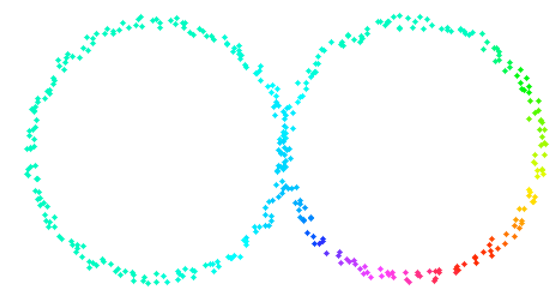
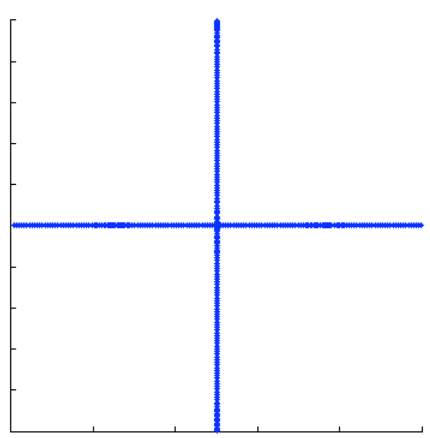
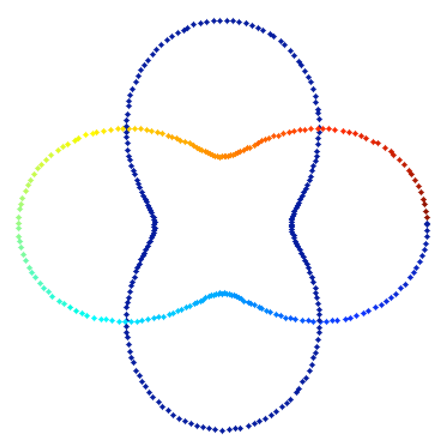
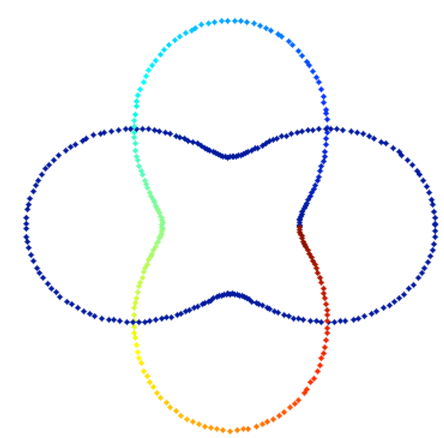
Rotating cube images



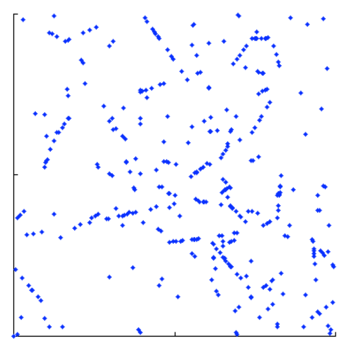
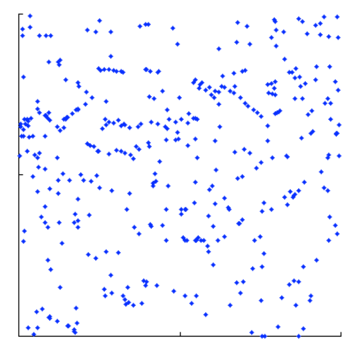
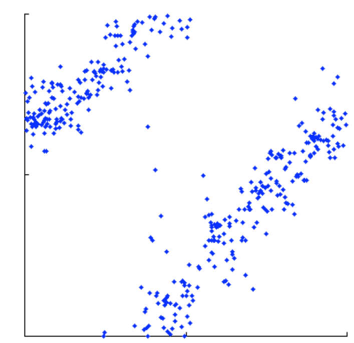
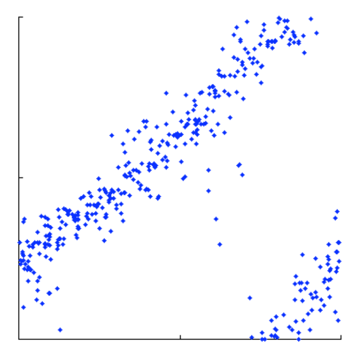
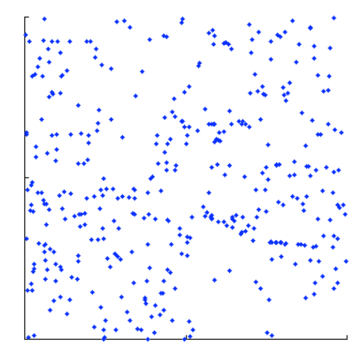
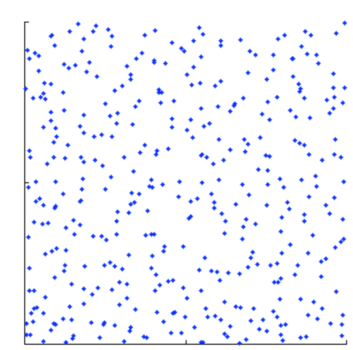
inferred vs original	histogram
	

Vin de Silva
http://pages.pomona.edu/~vds04747/

Pairs of circles

correlation	inferred_1	inferred_2
		
		

Noisy torus

	inferred$_2$	original$_1$	original$_2$
inferred$_1$			
inferred$_2$			
original$_1$			

Vin de Silva
http://pages.pomona.edu/~vds04747/

Persistent relative cohomology

Persistent homology

- Persistent homology algorithm (ELZ2000):
  - Given filtered simplicial complex $\{K_t, \rightarrow\}$
  - Input $(S, \partial S)$ in order of appearance of S
  - Output persistent homology

$$\{H_*(K_t), \rightarrow\}$$

- What happens if you input the cells in reverse order?

Persistent homology

- Persistent homology algorithm (ELZ2000):
  - Given filtered simplicial complex $\{K_t, \to\}$
  - Input $(S, \partial S)$ in order of appearance of S

    Note: When S enters the filtration, the simplices of ∂S are already there.
  - Output persistent homology

    $$\{H_*(K_t), \to\}$$
- What happens if you input the cells in reverse order?

Persistent relative cohomology

- Persistent homology algorithm (ELZ2000):
  - Given filtered simplicial complex $\{K_t, \to\}$
  - Input $(S, \delta S)$ in order of appearance of S

    When S enters the reversed filtration, the simplices of δS are already there.
  - Output persistent relative cohomology

    $$\{H^*(K,K_t), \to\}$$

    (at time t, the missing cells are those of K_t).
- Next: exploit this for local homology calculations.

Local cohomology

- Local structure of X near a point x_0 measured by $H^*(X, X-x_0)$.
- Filtration $\mathbf{X} = \{X_t\}_{t<0}$ converging to $X-x_0$ as $t \to 0$ from below:

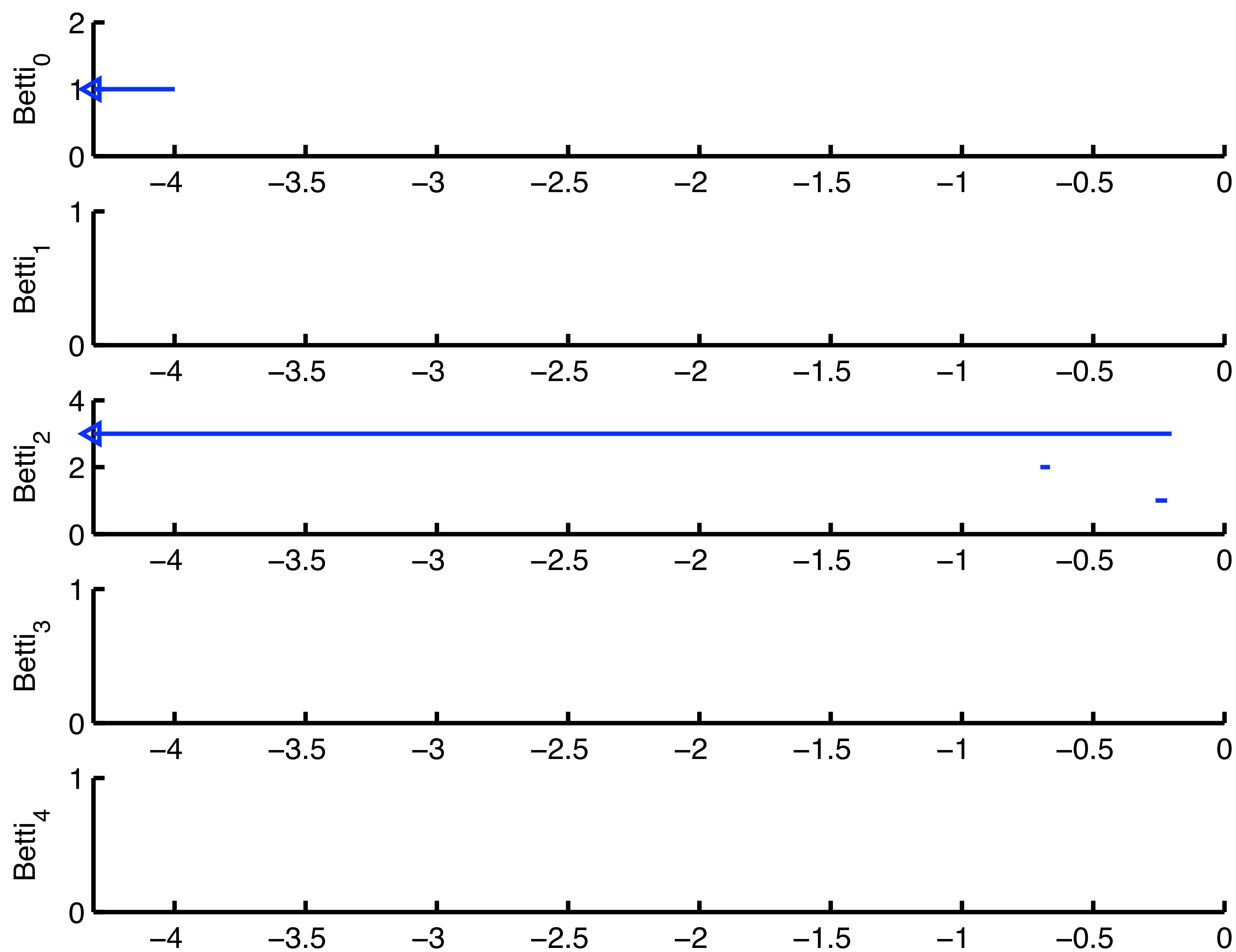
  $$X_t = \{x \in X : d(x,x_0) > |t|\}$$

- Restrict filtration to data points:

  $$\mathbf{B} = \{B_t = B \cap X_t\}_{t<0}$$

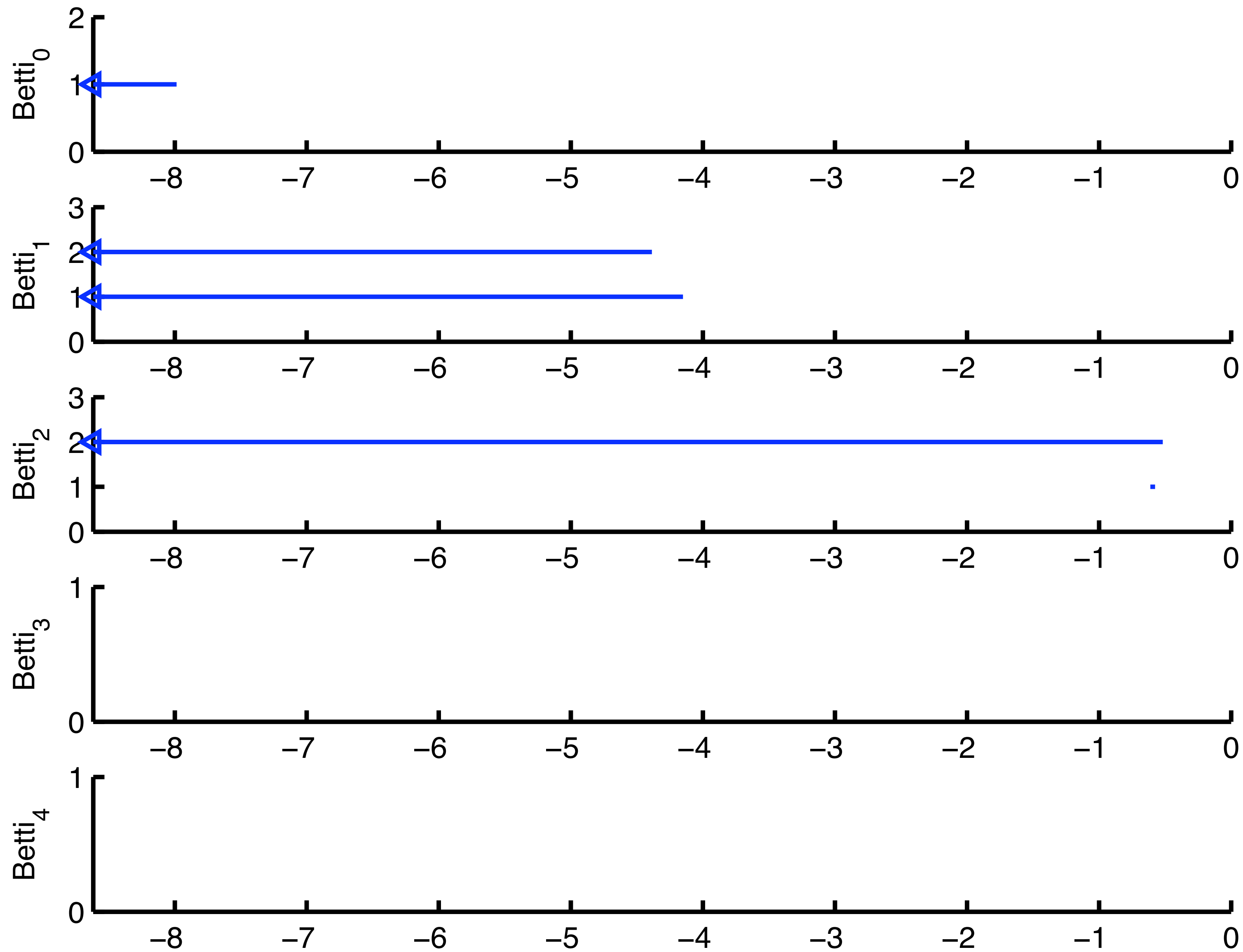
- Select landmarks $A \subset B$
- Fix ϵ
  - Construct filtered space $\mathbf{Del}(A,\mathbf{B};\epsilon) = \{Del^w(A,B_t;\epsilon)\}_{t<0}$.
  - Compute $H^*(\mathbf{Del}(A,\mathbf{B};\epsilon)) = pH^*(Del^w(A,B_t;\epsilon))$.

2-sphere



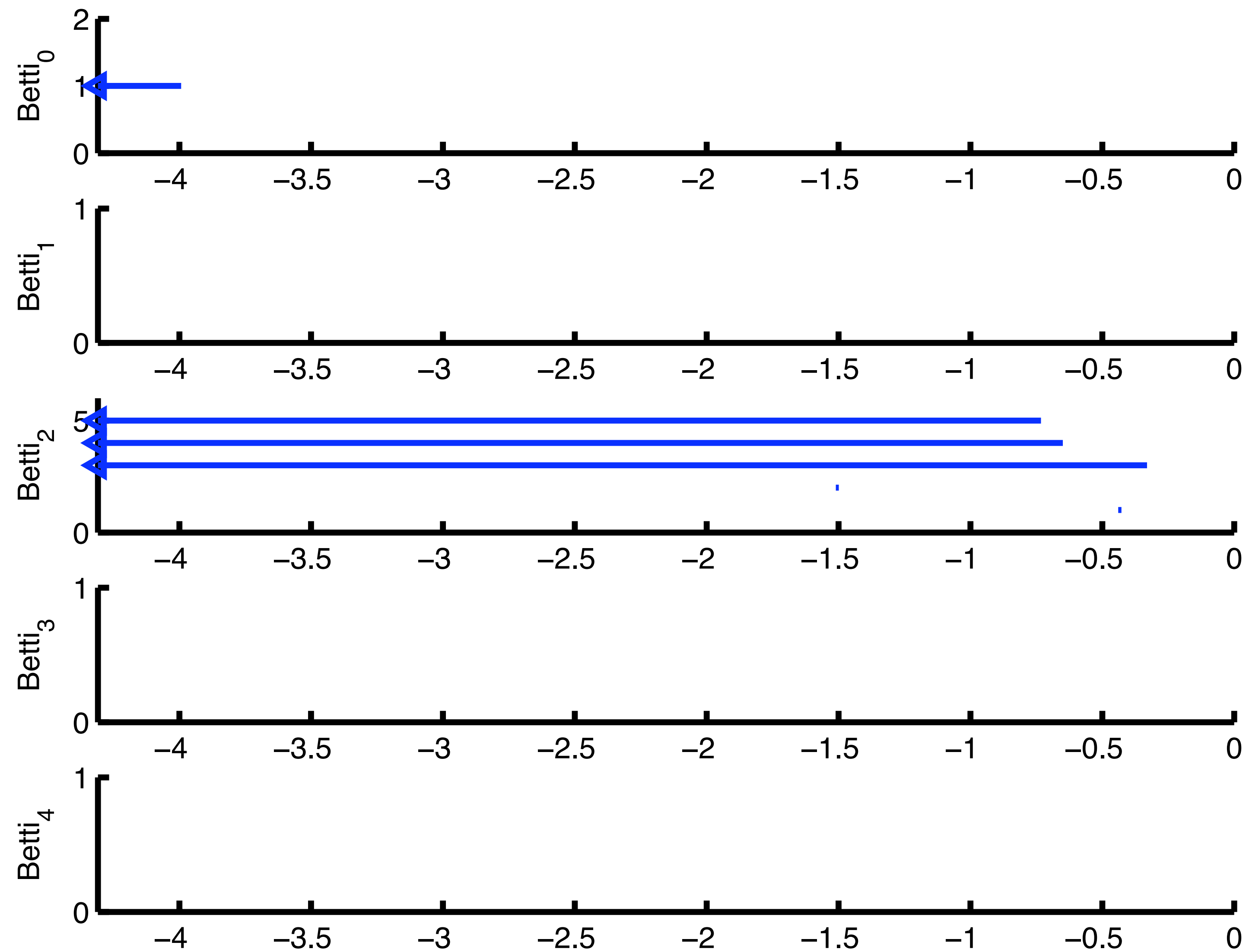
Betti_0
2
1
0
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
Betti_1
1
0
Betti_2
4
2
0
Betti_3
1
0
Betti_4
1
0

Torus

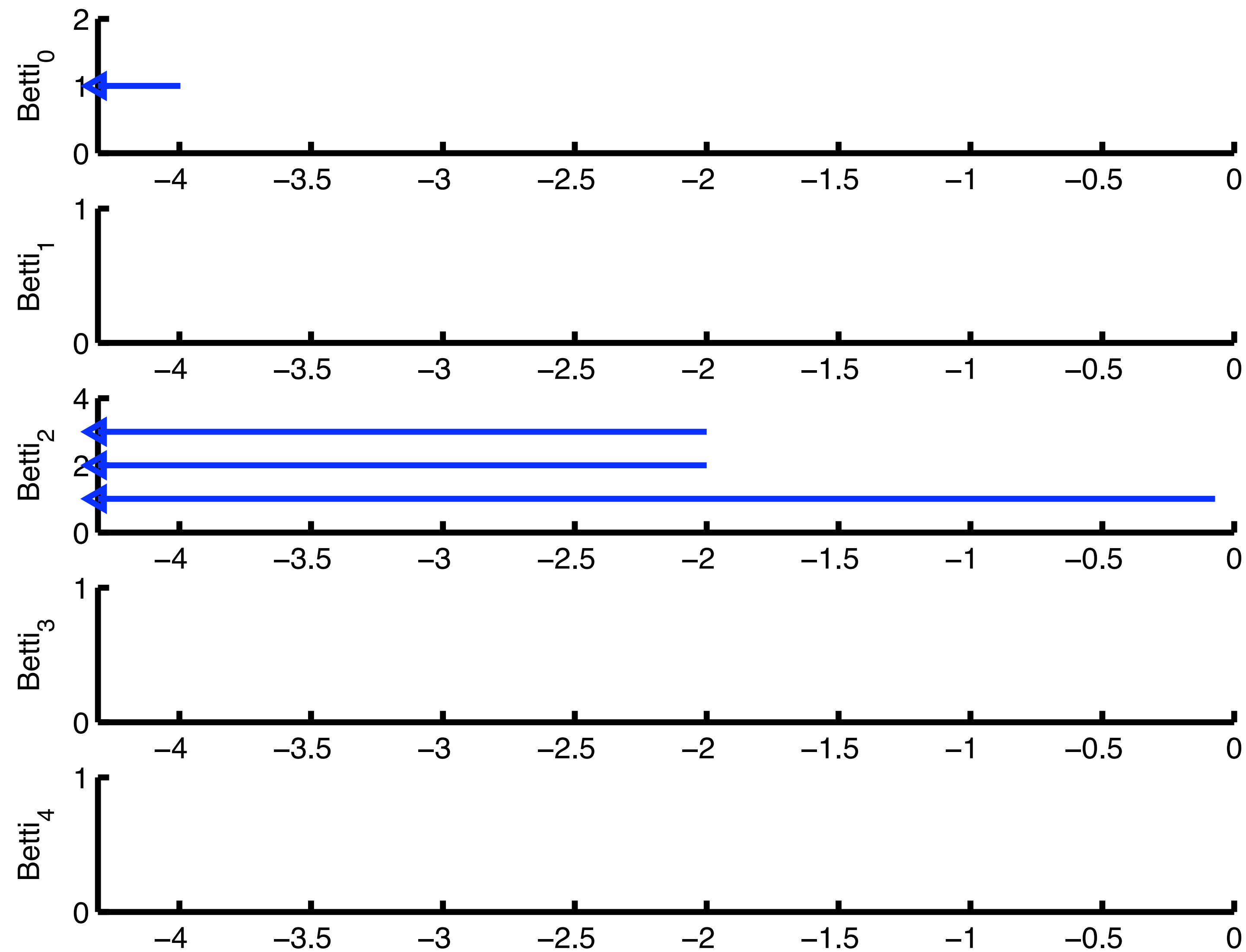


Betti$_0$
2
1
0
−8
−7
−6
−5
−4
−3
−2
−1
0
Betti$_1$
3
2
1
0
Betti$_2$
3
2
1
0
Betti$_3$
1
0
Betti$_4$
1
0

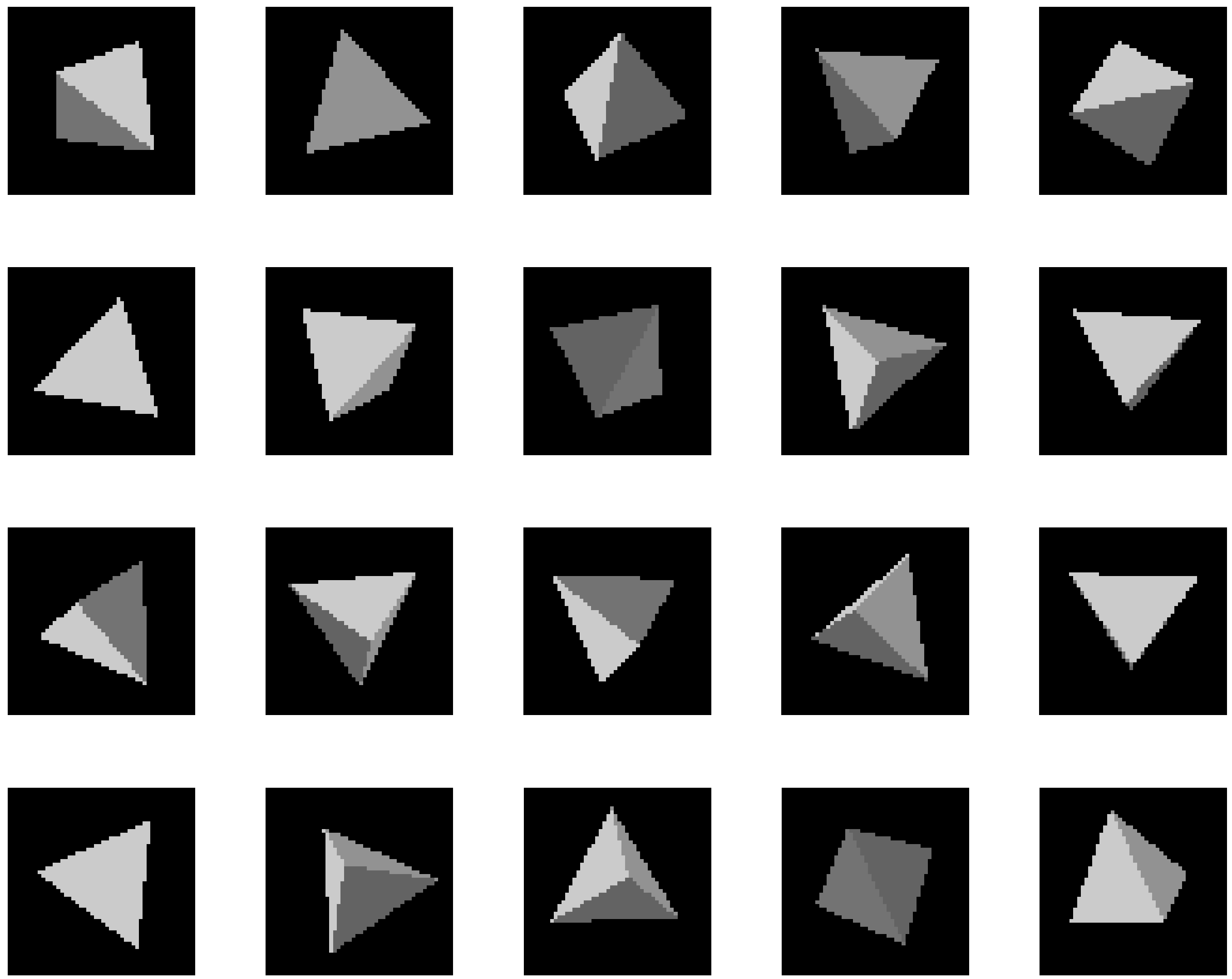
Union of two 2-spheres over a circle



Union of two 2-spheres over a circle

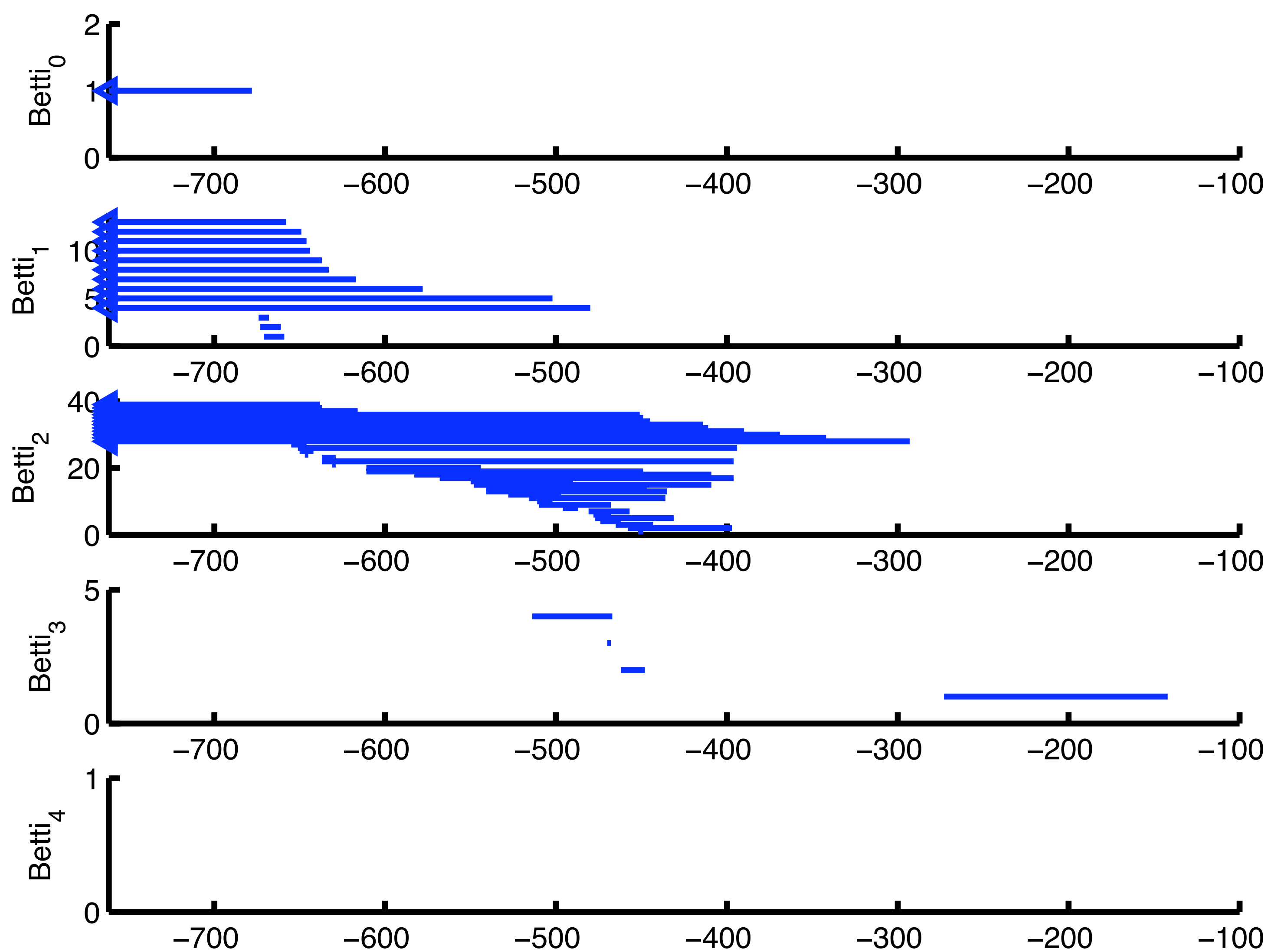


Space of tetrahedron images

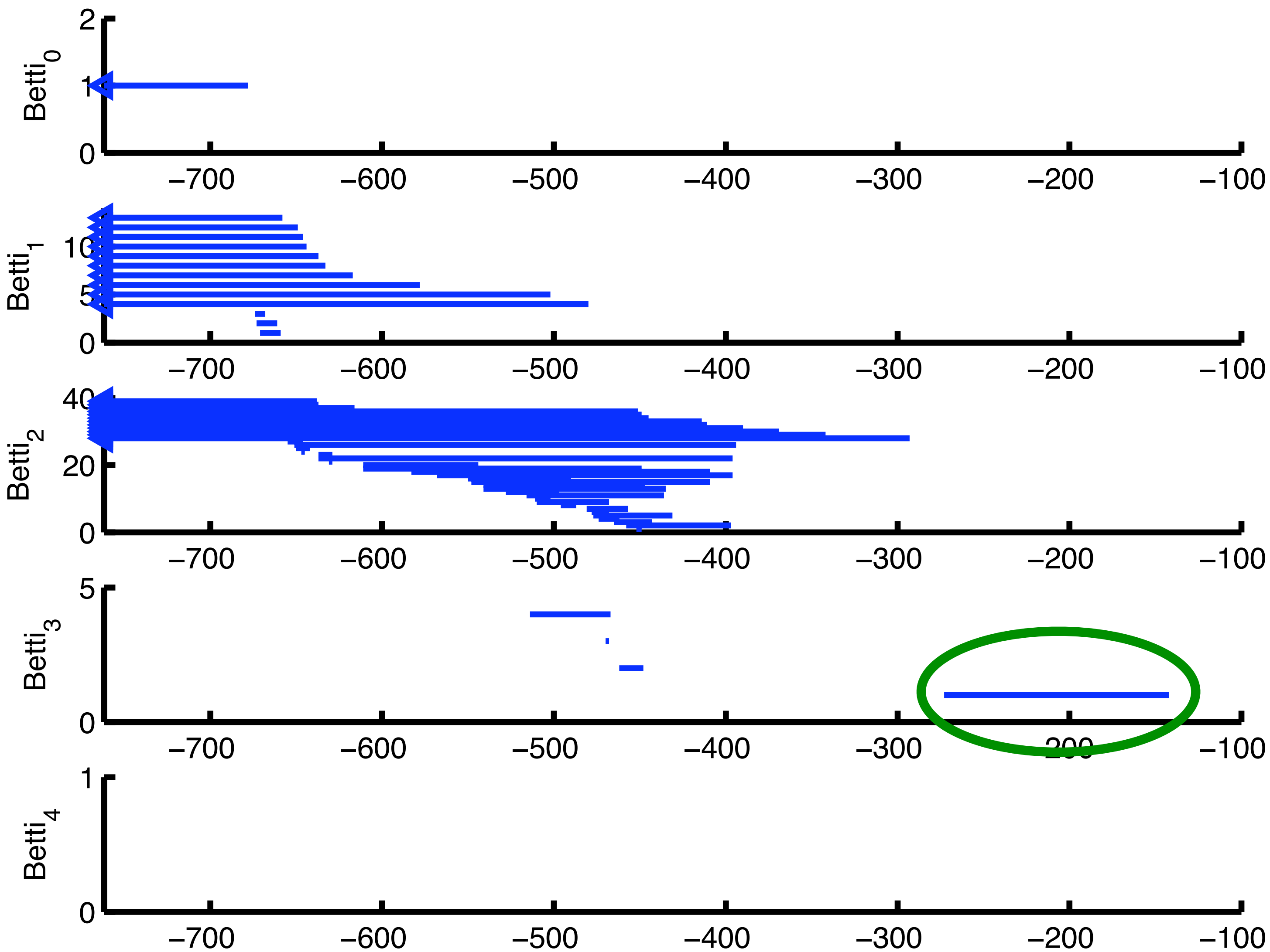


Space of rotations = SO(3) is 3-dimensional

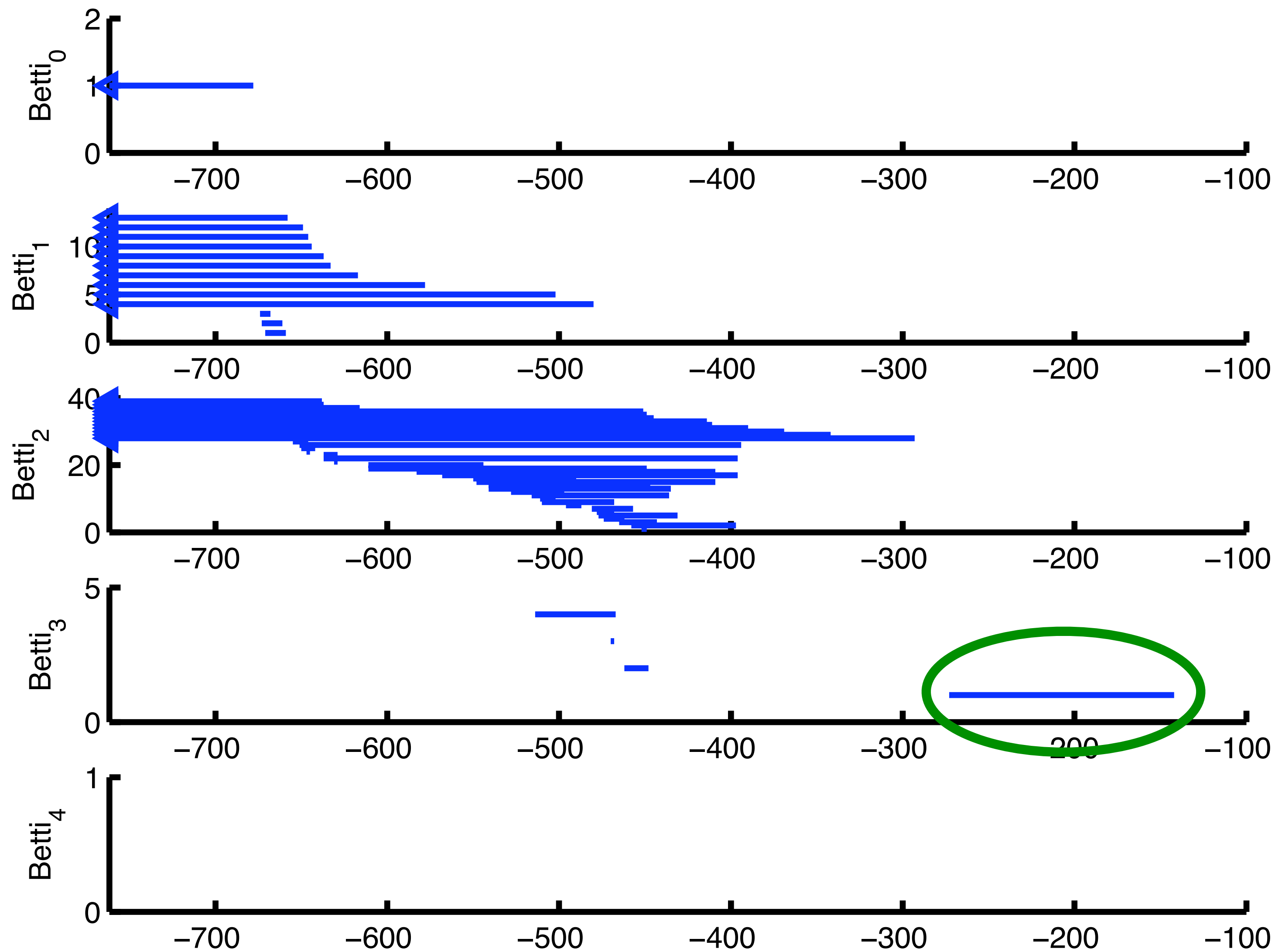
Space of tetrahedron images



Space of tetrahedron images



Space of tetrahedron images



Confession: I cheated a bit (by cherry-picking)

Zigzag persistence

Joint work with Gunnar Carlsson, Dmitriy Morozov

Three parameters

Persistence

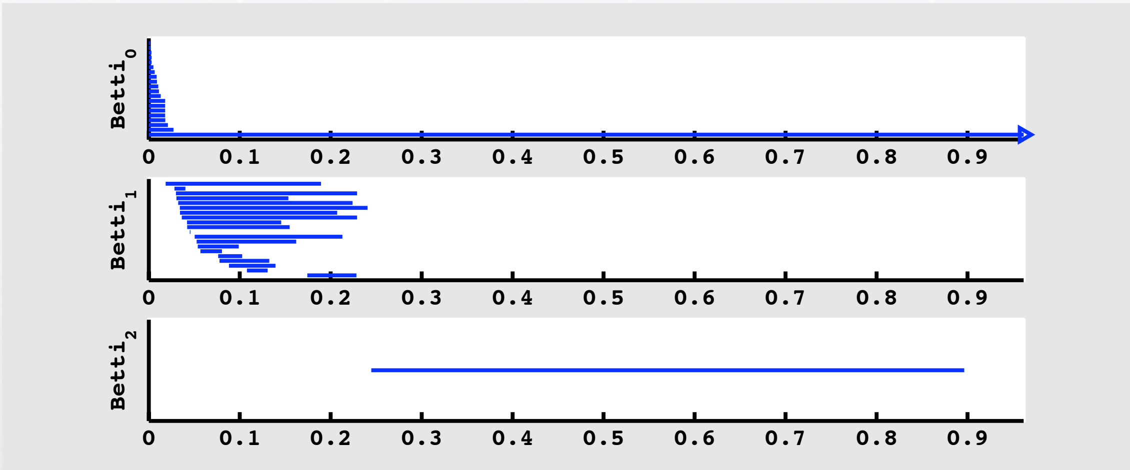
- Monotone increasing family of spaces

$$\mathbf{X} = \{X_\epsilon \mid \epsilon \geq 0\} \quad \text{such that} \quad X_\epsilon \subseteq X_{\epsilon'} \text{ if } \epsilon \leq \epsilon'$$

- Persistent homology

$$\operatorname{rank}\left[H_*(X_\epsilon) \to H_*(X_{\epsilon'})\right] \quad \text{for all } \epsilon \leq \epsilon'$$

- Barcode description (Edelsbrunner, Letscher, Zomorodian '00)
- Barcode stability theorem (Cohen-Steiner, Edelsbrunner, Harer '07)
  - the barcode depends continuously on the underlying diagram of spaces
  - see also Chazal, Cohen-Steiner, Glisse, Guibas, Oudot '09



Persistence

- Spaces

$$X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n$$

- Persistent homology

$$H_*(X_1) \longrightarrow H_*(X_2) \longrightarrow \cdots \longrightarrow H_*(X_n)$$

- Barcode algebra (Carlsson, Zomorodian '05)
  - H(**X**) is naturally a module over polynomial ring k[t]
  - ⇒ has unique representation as a sum of indecomposables
  - indecomposable summands depicted as barcode intervals
  - calculate decomposition using linear algebra over k[t]

Decomposition into summands

- Examples of indecomposable summands

$$0 \longrightarrow 0 \longrightarrow k \xrightarrow{\mathrm{Id}} k \xrightarrow{\mathrm{Id}} k \longrightarrow 0$$

$$0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

- Example of decomposable system

$$0 \longrightarrow k \xrightarrow{f} k^2 \xrightarrow{g} k \longrightarrow 0 \longrightarrow 0$$

- (decomposition depends on f,g)

Decomposition into summands

- Examples of indecomposable summands

$$0 \longrightarrow 0 \longrightarrow k \xrightarrow{\mathrm{Id}} k \xrightarrow{\mathrm{Id}} k \longrightarrow 0$$

$$0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

- Example of decomposable system

$$0 \longrightarrow k \xrightarrow{f} k^2 \xrightarrow{g} k \longrightarrow 0 \longrightarrow 0$$

- (decomposition depends on f,g)

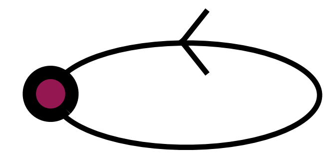
Non-monotone families

- Example: time-varying complex X = {X_t | t real}.
- Example: 30% strained data soup, varying the smoothing parameter.
- Example: witness complex with fixed vertex set, varying the set of witnesses.
- How do the features of X change as t varies?
  - New cell appears $X \longrightarrow X \cup \sigma$
  - Old cell disappears $X \longleftarrow X \setminus \tau$
  - Inclusion map directions vary arbitrarily, e.g.

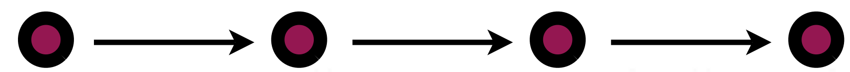
$$\cdots \longrightarrow X_{i-1} \longleftarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

- Can we do non-monotone persistence?

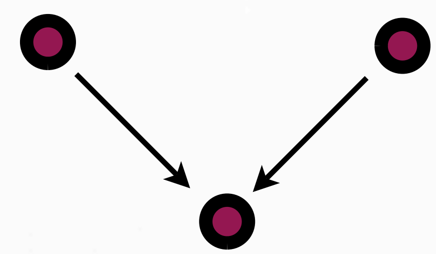
Quivers

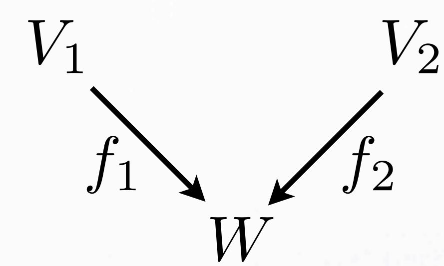


- A **quiver** is a directed (multi-)graph:
  - nodes
  - arrows



- A **representation** of a quiver Q has:
  - a vector space for every node
  - a linear map for every arrow





- **General question**: classify representations of a given quiver Q.
- What would be the ideal answer?
  - unique decomposition into indecomposable representations
  - + explicit list of indecomposables
  - + algorithm to determine decomposition type

Quivers

- Example ●
- Typical representation V
- Irreducible representations (over complex numbers)

  $\mathbb{C}$

- Classifying invariant $\dim(V)$

Quivers

- Example

- Typical representation

$$V \xrightarrow{f} W$$

- Irreducible representations (over complex numbers)

$$\mathbb{C} \xrightarrow{1} \mathbb{C}$$

$$\mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C}$$

- Classifying invariants $\mathrm{rank}(f),\ \dim(V),\ \dim(W)$

Quivers

- Example

- Typical representation

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$$

- Irreducible representations (over complex numbers)

$$\text{intervals } [b, d], \quad 0 \leq b \leq d \leq 3$$

- Classifying invariants

$$\text{rank}\, [V_i \to V_j]\,, \quad 0 \leq i \leq j \leq 3$$

Quivers

- Example

- Typical representation

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$$

- Irreducible representations (over complex numbers)

$$\text{intervals } [b, d], \quad 0 \leq b \leq d \leq 3$$

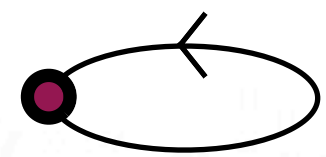
- Classifying invariants

$$\text{rank}\left[V_i \to V_j\right], \quad 0 \leq i \leq j \leq 3$$

relation: $\text{rank}\,[V_i \to V_j] = \sum_{[b,d] \supseteq [i,j]} \text{multiplicity of } [b, d]$

Quivers

- Example



- Typical representation

$$V \xrightarrow{f} V$$

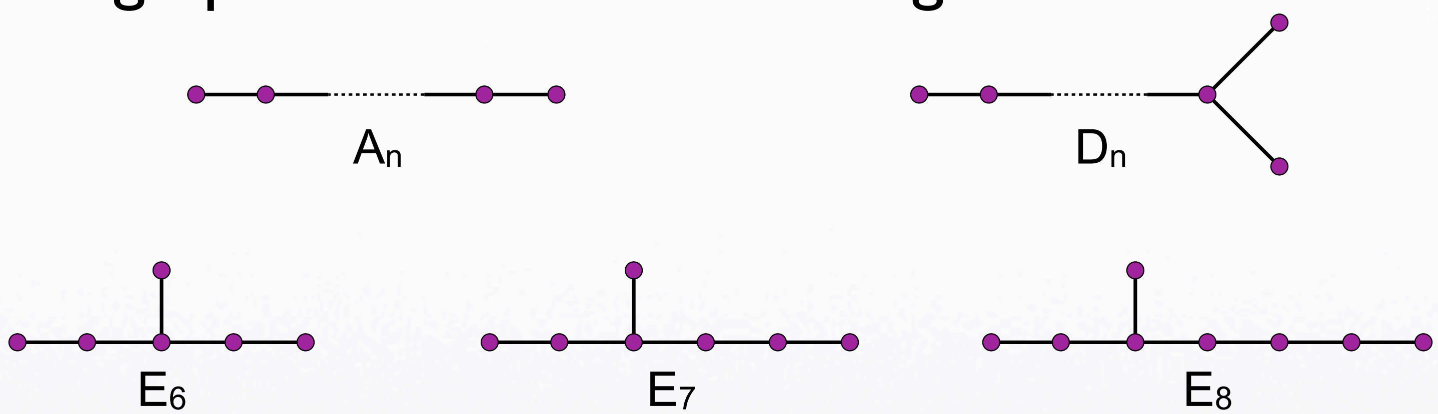
- Irreducible representations (over complex numbers)

$$\text{Jordan blocks} \begin{bmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

- Classifying invariants: generalised eigenspectrum of f

Quivers

- A quiver Q is of finite type if there is a unique decomposition theorem with a finite list of indecomposables.
- Gabriel's Theorem (1972): Q is of finite type iff its underlying undirected graph is one of the following.



- Kac's theorem (1980): the set of dimension vectors of indecomposable representations of a Q is independent of the direction of the arrows.

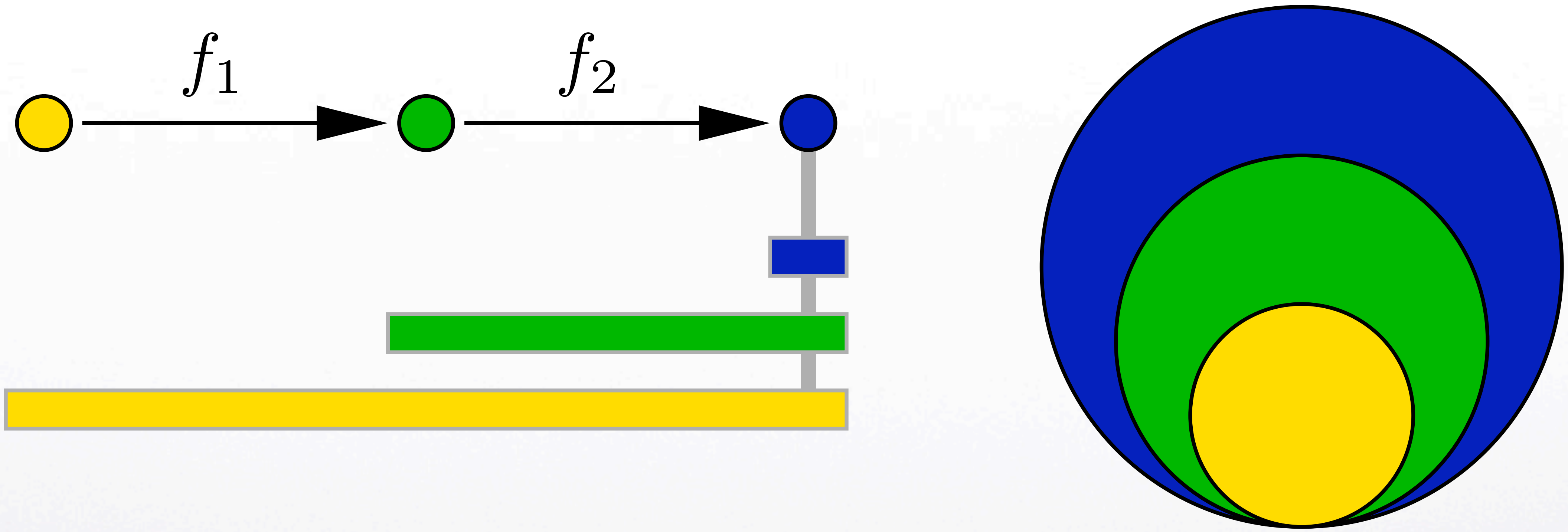
Corollary: interval decomposition for all quivers of type A_n

⇒ zigzag persistent homology!

Calculating the zig-zag barcode

- We can construct the barcode representation inductively.
  - Start from the left term in the sequence.
  - At time k, we have a filtration of V_k stored as a filtered basis.
  - The filtration records the lifetimes (to date) of all vectors in V_k.
  - Update step: pull back/push forward the filtration to V_{k+1}.
  - Record the lifespans of features killed in the move.
- The filtration is crucial. Basis operations must respect the filtration.

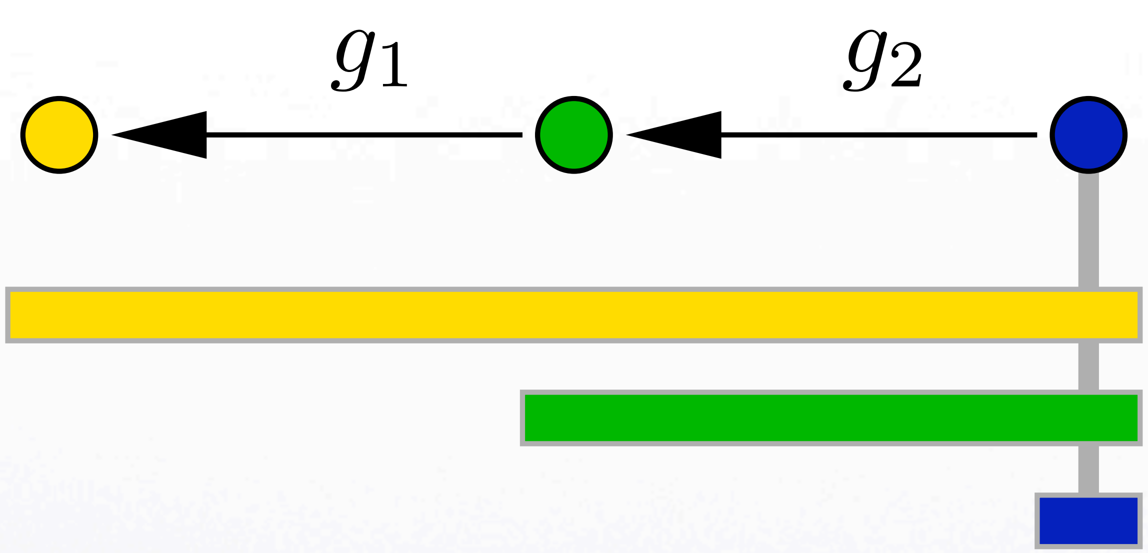
Three-term sequences

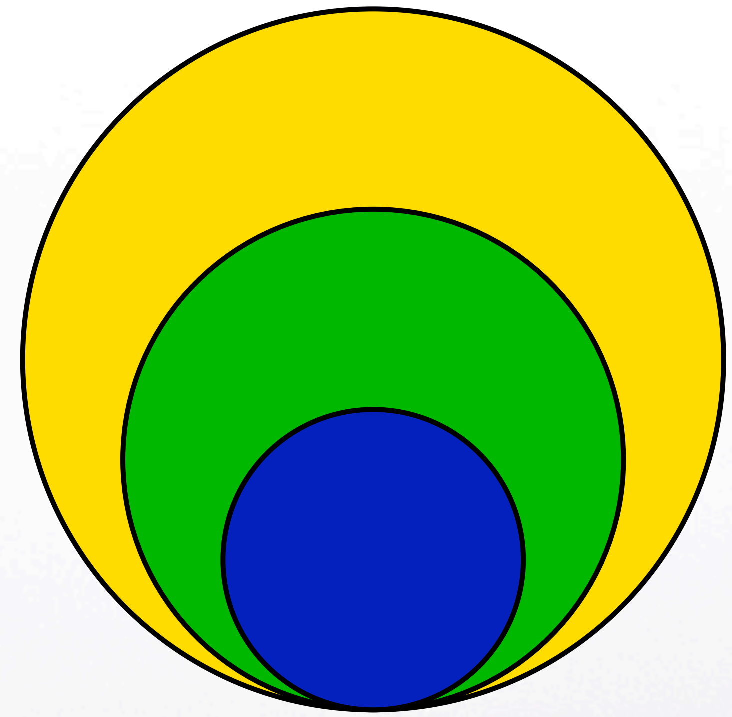


$$f_2(f_1(V_0)) \subseteq f_2(V_1) \subseteq V_2$$

$* \quad * \quad *$

Three-term sequences

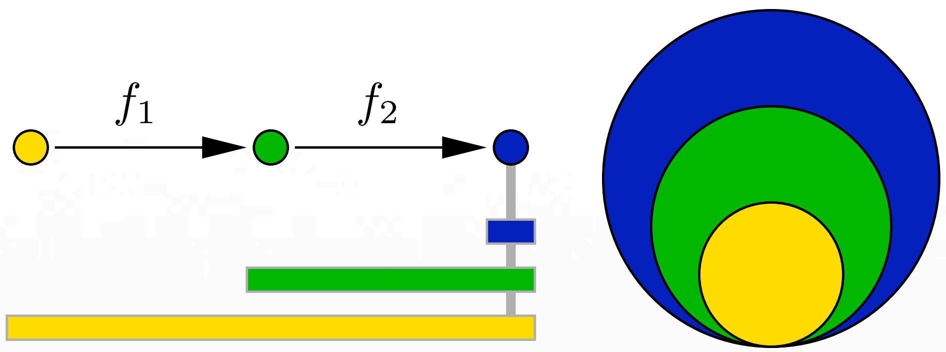




$$g_2^{-1}(0) \subseteq g_2^{-1}(g_1^{-1}(0)) \subseteq V_2$$

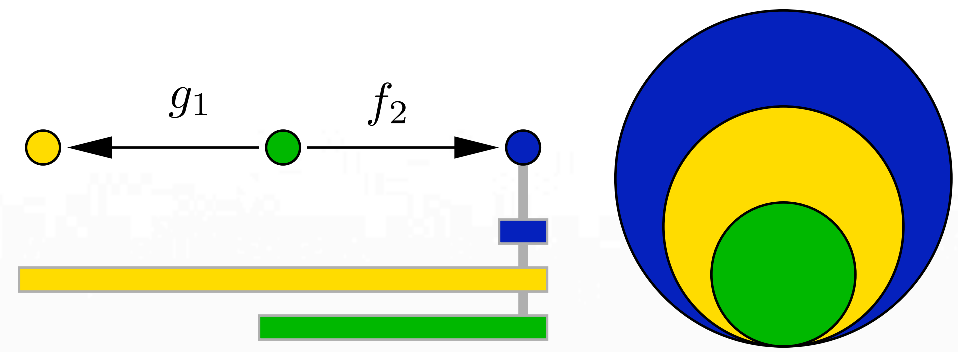
* * *

Three-term sequences



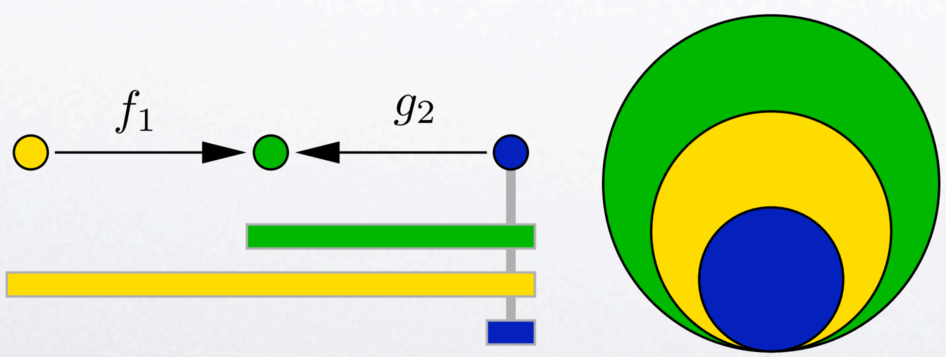
$f_2(f_1(V_0)) \subseteq f_2(V_1) \subseteq V_2$

$* \quad * \quad *$



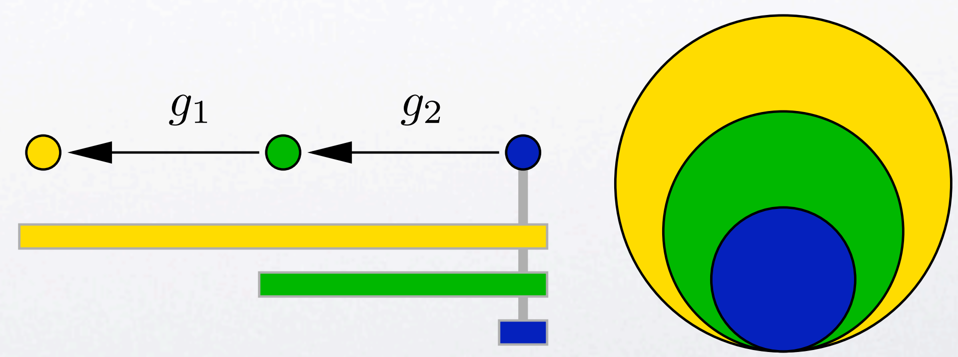
$f_2(g_1^{-1}(0)) \subseteq f_2(V_1) \subseteq V_2$

$* \quad * \quad *$



$g_2^{-1}(0) \subseteq g_2^{-1}(f_1(V_0)) \subseteq V_2$

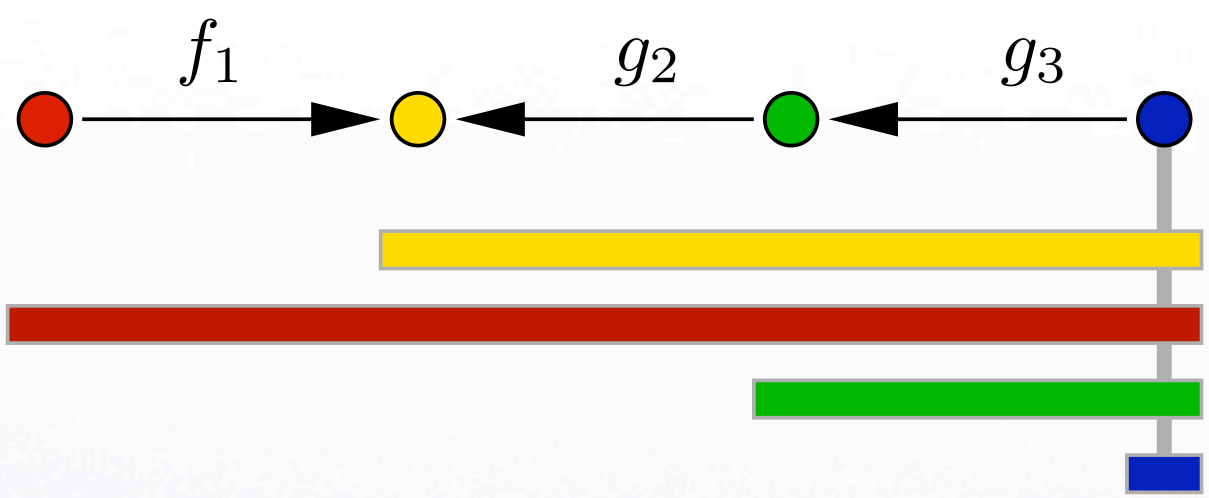
$* \quad * \quad *$

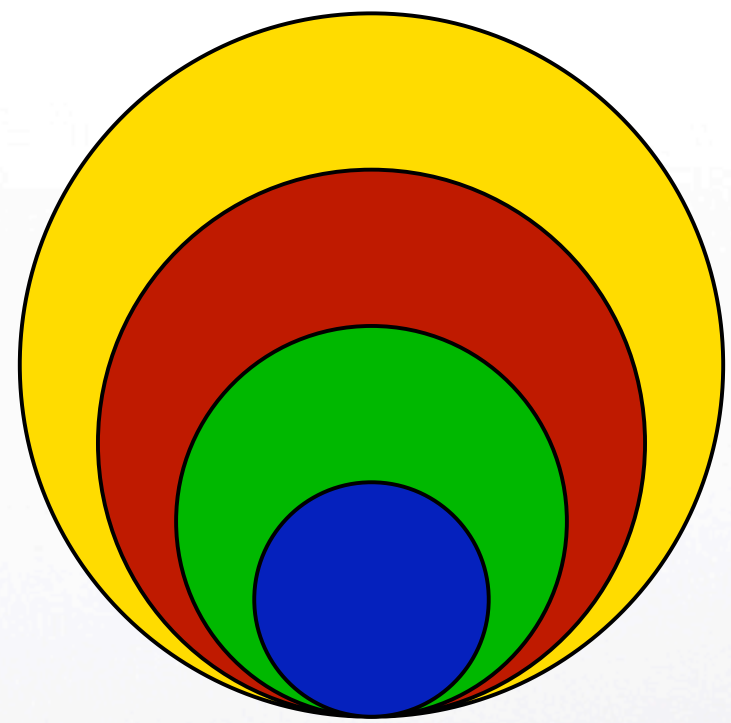


$g_2^{-1}(0) \subseteq g_2^{-1}(g_1^{-1}(0)) \subseteq V_2$

$* \quad * \quad *$

A four-term sequence





$$g_3^{-1}(0) \subseteq g_3^{-1}(g_2^{-1}(0)) \subseteq g_3^{-1}(g_2^{-1}(f_1(V_0))) \subseteq V_3$$

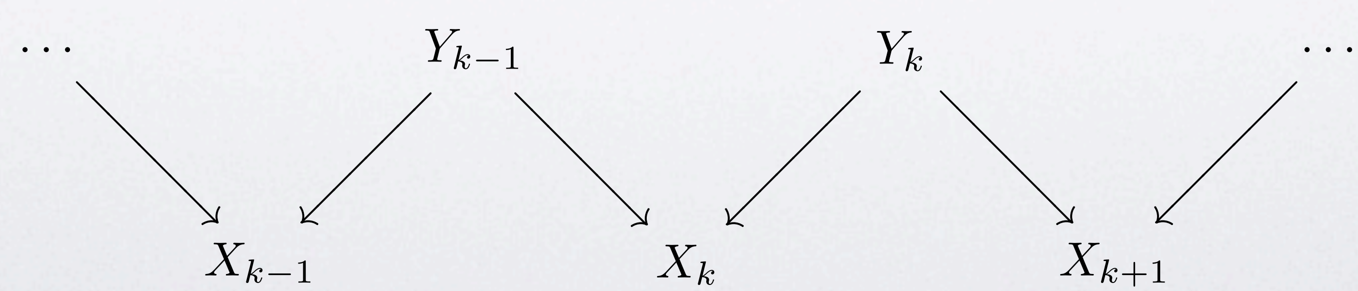
* * * *

Witness bicomplexes

Joint work with Gunnar Carlsson, Dmitriy Morozov

Families of witness complexes

- X_k = Del(A,C) where C = C(t_k)
  - Vertex set fixed, add or subtract cells one at a time
  - H(X_k) is a quiver representation of type A_n
  - A_n-quiver decomposition ⇒ interval barcode ✓
- X_k = Del(A,C) where A = A(t_k)
  - No natural map H(X_k) → H(X_{k+1}) or H(X_k) ← H(X_{k+1})
  - Construct interpolating spaces Y_k with maps $X_k \leftarrow Y_k \rightarrow X_{k+1}$
  - Apply A_n-quiver decomposition to zigzag sequence ✓



Witness bicomplexes

- A, B, X subsets of a metric space
- Strong 2-witnesses

$$\begin{aligned} x \in X &\text{ is a strong biwitness for bisimplex } (\sigma, \tau) \\ \Leftrightarrow &\quad x \text{ is a strong witness for } \sigma \subset A \\ \text{AND} &\quad x \text{ is a strong witness for } \tau \subset B \end{aligned}$$

- Weak 2-witnesses

$$\begin{aligned} x \in X &\text{ is a weak biwitness for bisimplex } (\sigma, \tau) \\ \Leftrightarrow &\quad x \text{ is a weak witness for } \sigma \subset A \\ \text{AND} &\quad x \text{ is a weak witness for } \tau \subset B \end{aligned}$$

- Strong Delaunay bicomplex

$$(\sigma, \tau) \in \mathrm{Del}_2(A, B; X) \quad \Leftrightarrow \quad (\sigma, \tau) \text{ has a strong biwitness } x \in X$$

- Weak Delaunay bicomplex

$$(\sigma, \tau) \in \mathrm{Del}_2^{\mathrm{w}}(A, B; X) \quad \Leftrightarrow \quad \text{every } (\sigma', \tau') \leq (\sigma, \tau) \text{ has a weak biwitness } x \in X$$

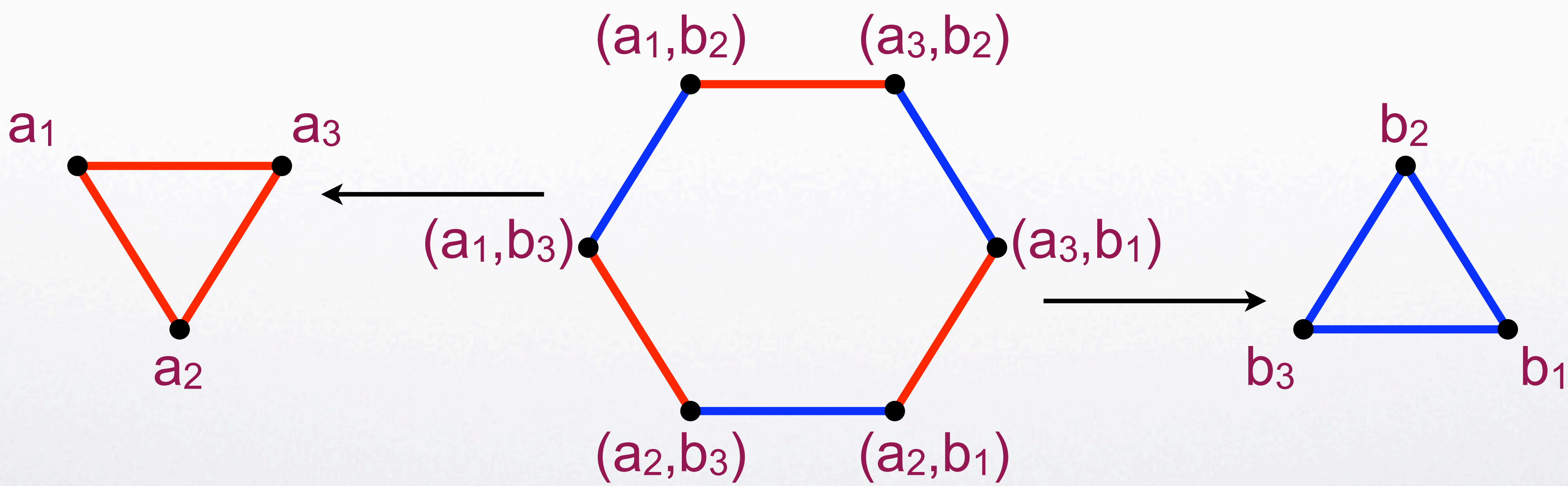
Witness bicomplexes

- There are natural projection maps

$$\mathrm{Del}(A;X) \longleftarrow \mathrm{Del}_2(A,B;X) \longrightarrow \mathrm{Del}(B;X)$$

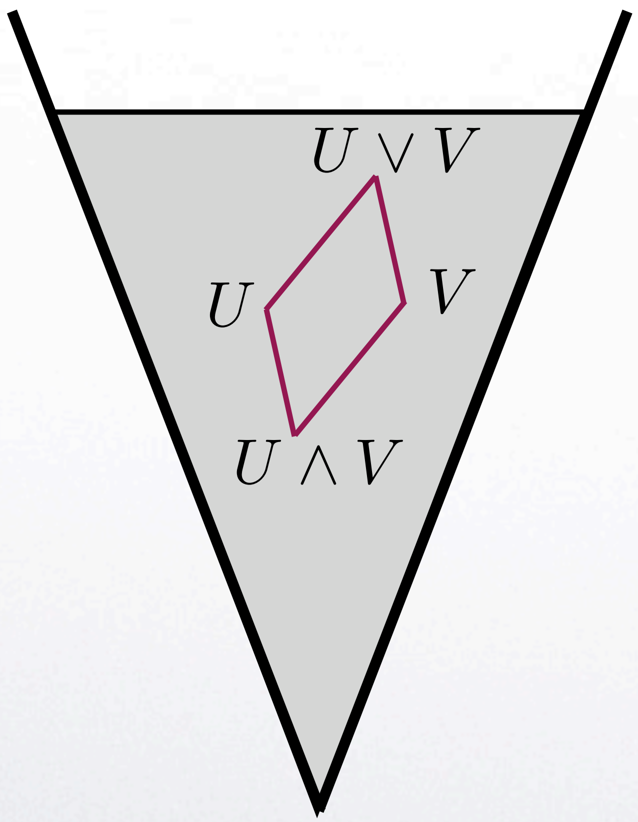
$$\mathrm{Del}^{\mathrm{w}}(A;X) \longleftarrow \mathrm{Del}_2^{\mathrm{w}}(A,B;X) \longrightarrow \mathrm{Del}^{\mathrm{w}}(B;X)$$

- Example: X = unit circle, A = {0, 2π/3, 4π/3}, B = {π/3, π, 5π/3}



- (To understand correctness, we consider 2-nerves.)

Binerve of a pair of covers



- Two covers of X

$$U = \{U_a \mid a \in A\}$$
$$V = \{V_b \mid b \in B\}$$

- Binerve

$$(\sigma, \tau) \in \mathcal{N}_2(U, V) \quad \Leftrightarrow \quad \bigcap_{a \in \sigma} U_a \cap \bigcap_{b \in \tau} V_b \neq \emptyset$$

- Union and intersection

$$U \vee V = \{U_a \mid a \in A\} \cup \{V_b \mid b \in B\}$$
$$U \wedge V = \{U_a \cap V_b \mid a \in A,\, b \in B\}$$

- Homotopy equivalence

$$\mathcal{N}_2(U, V) \simeq \mathcal{N}(U \wedge V)$$

- Mayer–Vietoris theorem

$$\ldots \to H_k(U \wedge V) \to H_k(U) \oplus H_k(V) \to H_k(U \vee V) \to H_{k-1}(U \wedge V) \to \ldots$$

The binerve theorem

- Proposition (weak witnesses theorem for Delaunay bicomplexes).
  - $\mathrm{Del}_2^{\mathrm{w}}(A, B; \mathbb{R}^n) = \mathrm{Del}_2(A, B; \mathbb{R}^n)$
- Proposition. Let U,V be the Voronoi covers of X defined by A,B.
  - $\mathrm{Del}_2(A, B; X) = \mathcal{N}_2(U, V)$
- Binerve Theorem. U,V covers of X. The following are equivalent.

$U \vee V$ is a projectively faithful cover of X

$\mathcal{N}(U) \leftarrow \mathcal{N}_2(U, V) \rightarrow \mathcal{N}(V)$ is projectively faithful* for X

$\mathcal{N}(U) \rightarrow \mathcal{N}(U \vee V) \leftarrow \mathcal{N}(V)$ is projectively faithful* for X

* i.e. the homology diagram contains H*(X) ← H*(X) → H*(X) as a summand

* i.e. the homology diagram contains H*(X) → H*(X) ← H*(X) as a summand

The pyramid theorem

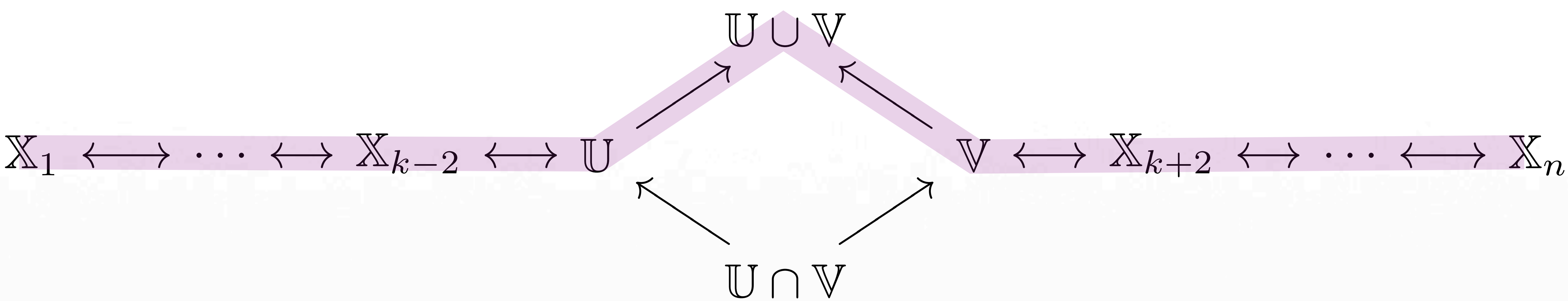
Joint work with Gunnar Carlsson, Dmitriy Morozov

Mayer–Vietoris Diamond Principle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \mathbb{U} \cup \mathbb{V} & & & & & & & \\
 & & & & & & \nearrow & & \nwarrow & & & & & & \\
\mathbb{X}_1 & \longleftrightarrow & \cdots & \leftrightarrow & \mathbb{X}_{k-2} & \longleftrightarrow & \mathbb{U} & & \mathbb{V} & \longleftrightarrow & \mathbb{X}_{k+2} & \leftrightarrow & \cdots & \longleftrightarrow & \mathbb{X}_n \\
 & & & & & & \nwarrow & & \nearrow & & & & & & \\
 & & & & & & & \mathbb{U} \cap \mathbb{V} & & & & & & &
\end{array}
$$

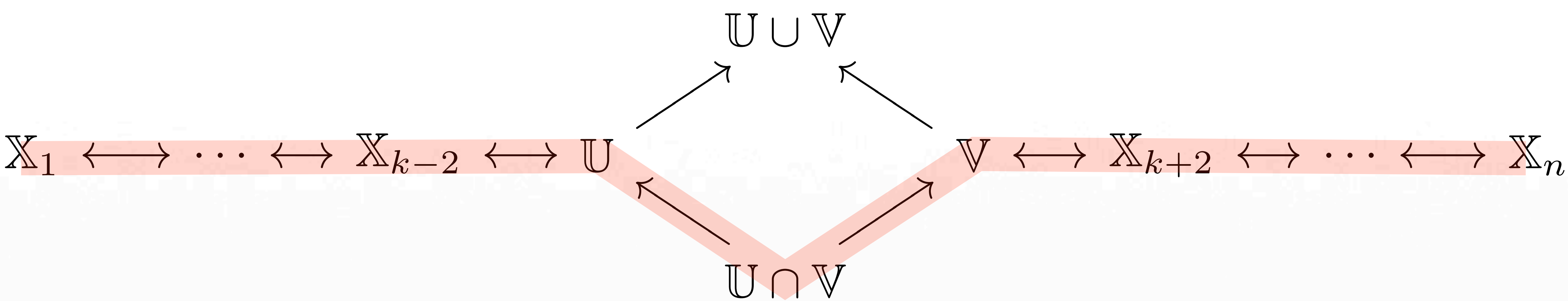
- The two zigzags in the above diagram carry the same persistent homology information.
  - The intervals change slightly, and some intervals undergo a dimension shift.
  - The result follows from the Mayer–Vietoris theorem for the middle diamond.

Mayer–Vietoris Diamond Principle



- The two zigzags in the above diagram carry the same persistent homology information.
  - The intervals change slightly, and some intervals undergo a dimension shift.
  - The result follows from the Mayer–Vietoris theorem for the middle diamond.

Mayer–Vietoris Diamond Principle



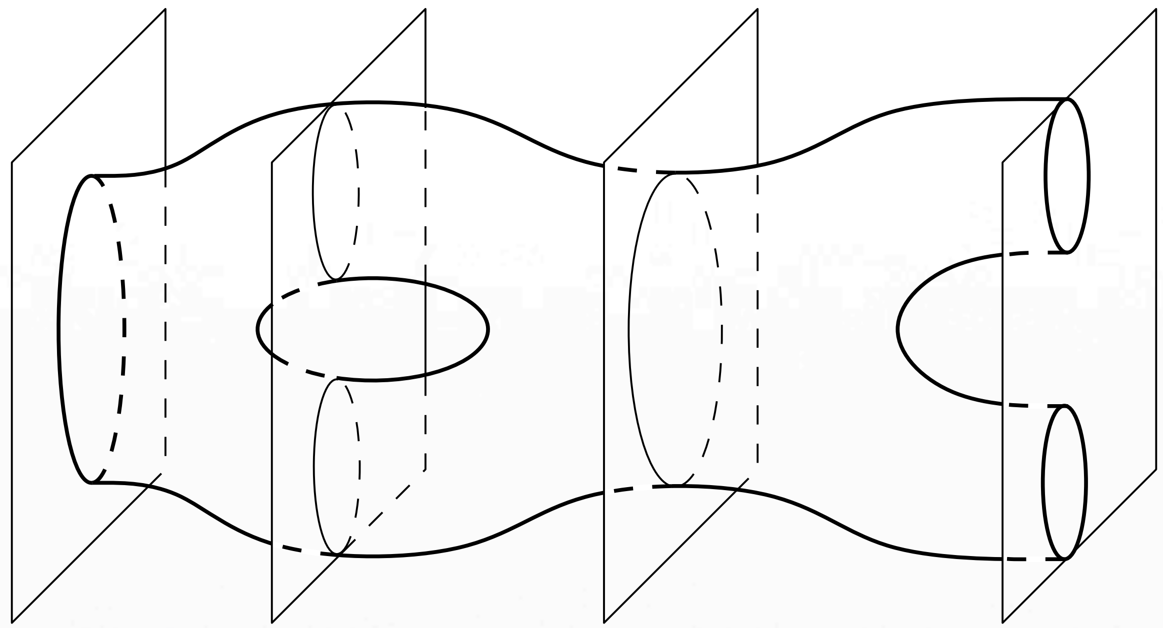
- The two zigzags in the above diagram carry the same persistent homology information.
  - The intervals change slightly, and some intervals undergo a dimension shift.
  - The result follows from the Mayer–Vietoris theorem for the middle diamond.

Mayer–Vietoris Diamond Principle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \mathbb{U} \cup \mathbb{V} & & & & & & & \\
 & & & & & & \nearrow & & \nwarrow & & & & & & \\
\mathbb{X}_1 & \longleftrightarrow & \cdots & \longleftrightarrow & \mathbb{X}_{k-2} & \longleftrightarrow & \mathbb{U} & & \mathbb{V} & \longleftrightarrow & \mathbb{X}_{k+2} & \longleftrightarrow & \cdots & \longleftrightarrow & \mathbb{X}_n \\
 & & & & & & \nwarrow & & \nearrow & & & & & & \\
 & & & & & & & \mathbb{U} \cap \mathbb{V} & & & & & & &
\end{array}
$$

- The two zigzags in the above diagram carry the same persistent homology information.
  - The intervals change slightly, and some intervals undergo a dimension shift.
  - The result follows from the Mayer–Vietoris theorem for the middle diamond.

Levelset zigzag persistence

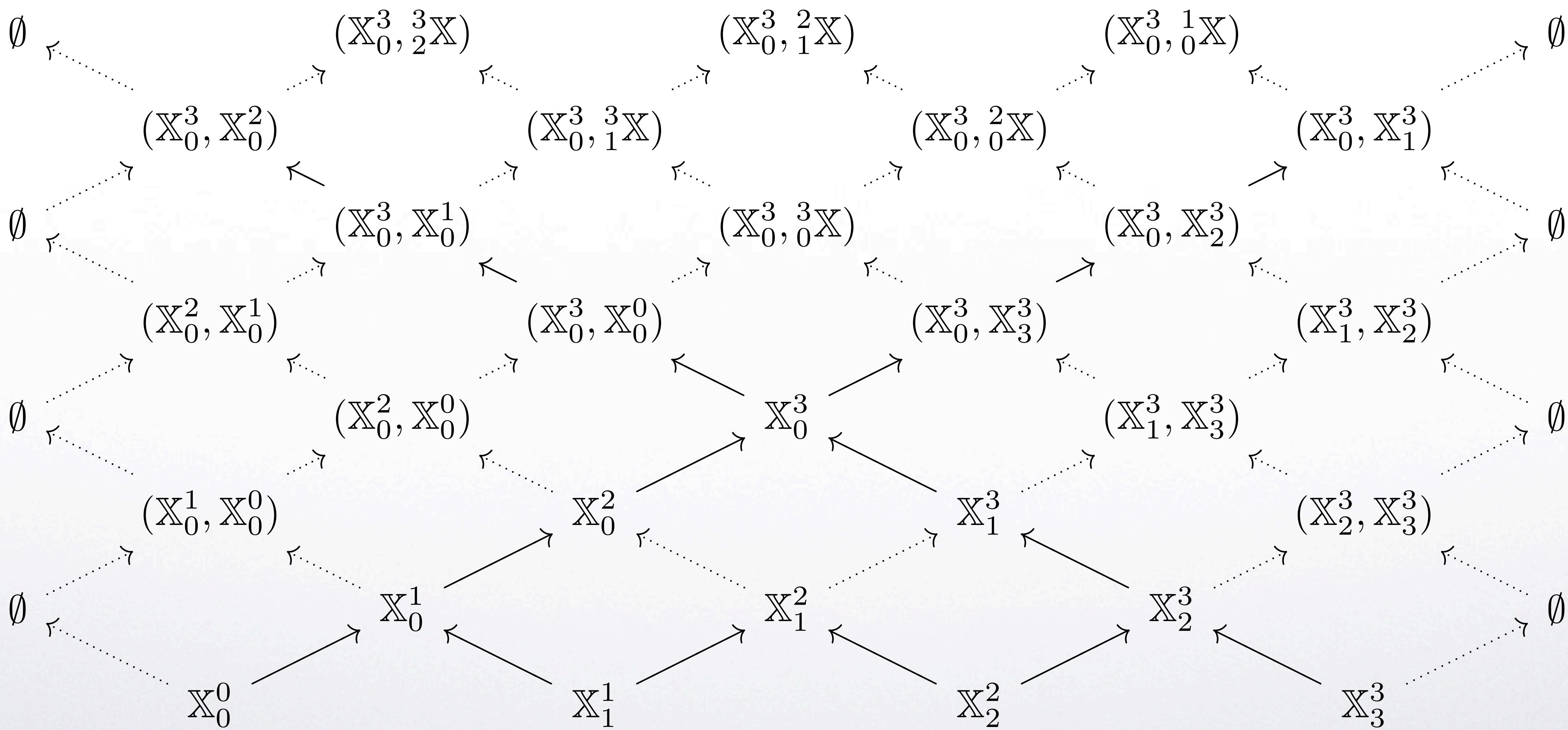


- X any space with a (tame) real-valued function f.
- Define the levelset zigzag of (X,f):

$$\mathbb{X}_0^0 \to \mathbb{X}_0^1 \leftarrow \mathbb{X}_1^1 \to \mathbb{X}_1^2 \leftarrow \mathbb{X}_2^2 \to \ldots \leftarrow \mathbb{X}_{n-1}^{n-1} \to \mathbb{X}_{n-1}^{n} \leftarrow \mathbb{X}_n^n,$$

$$\mathbb{X}_i^j = f^{-1}[i,j]$$

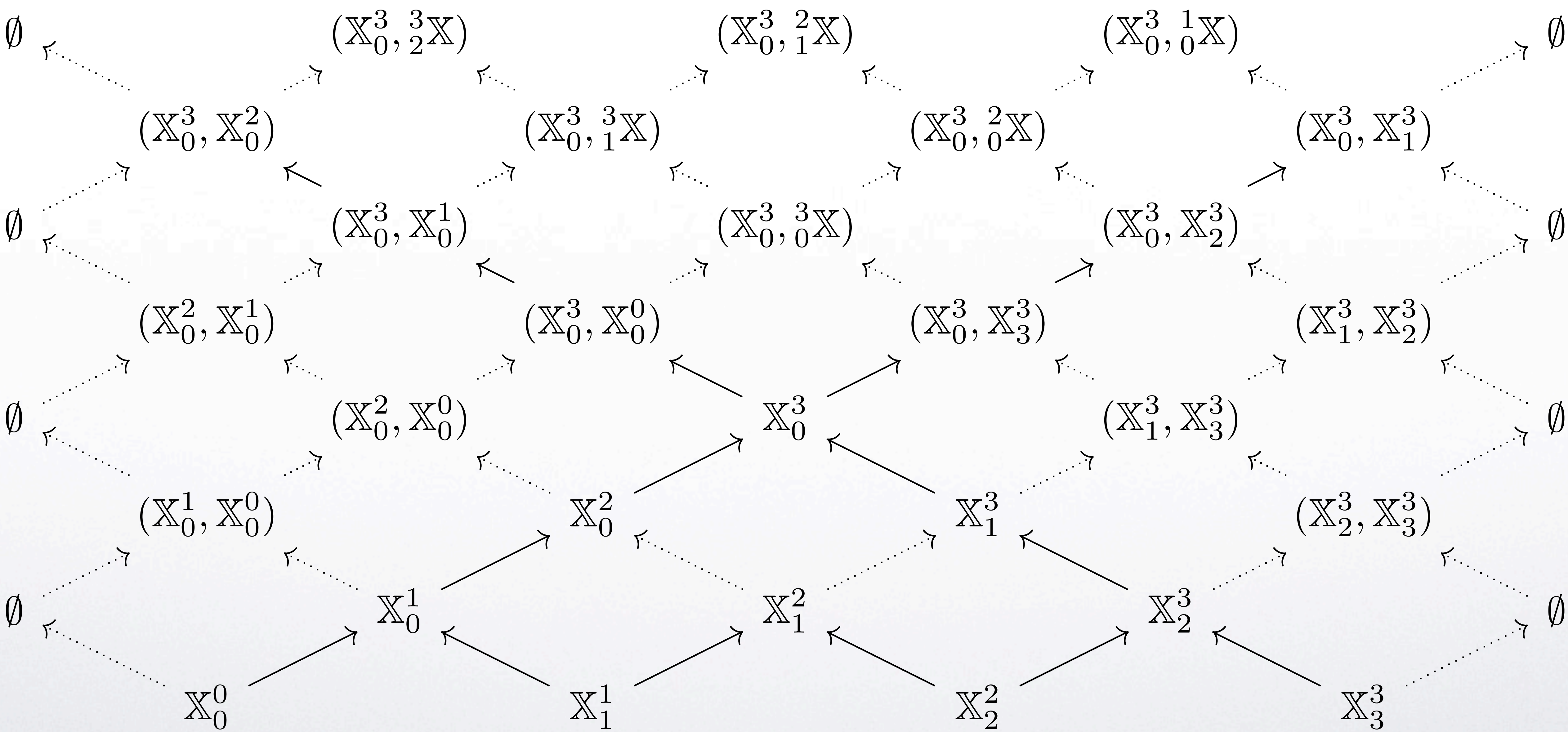
A vast commutative diagram



$\mathbb{X}_i^j = f^{-1}[i, j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

A vast commutative diagram

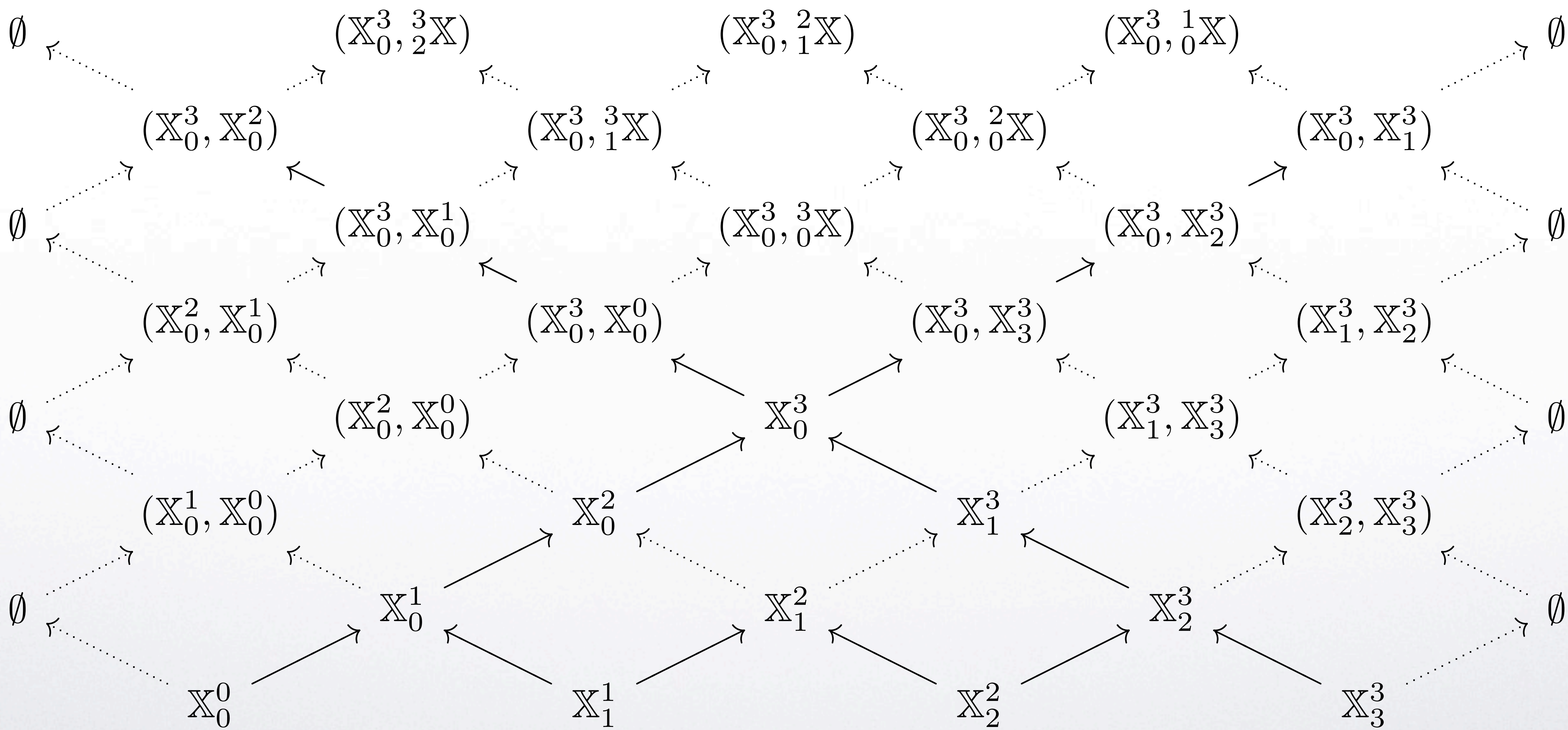


(In homology, there are connecting maps from the top edge to the bottom edge, turning this into a commutative Möbius band.)

$\mathbb{X}_i^j = f^{-1}[i, j]$

${}^j_i\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

A vast commutative diagram

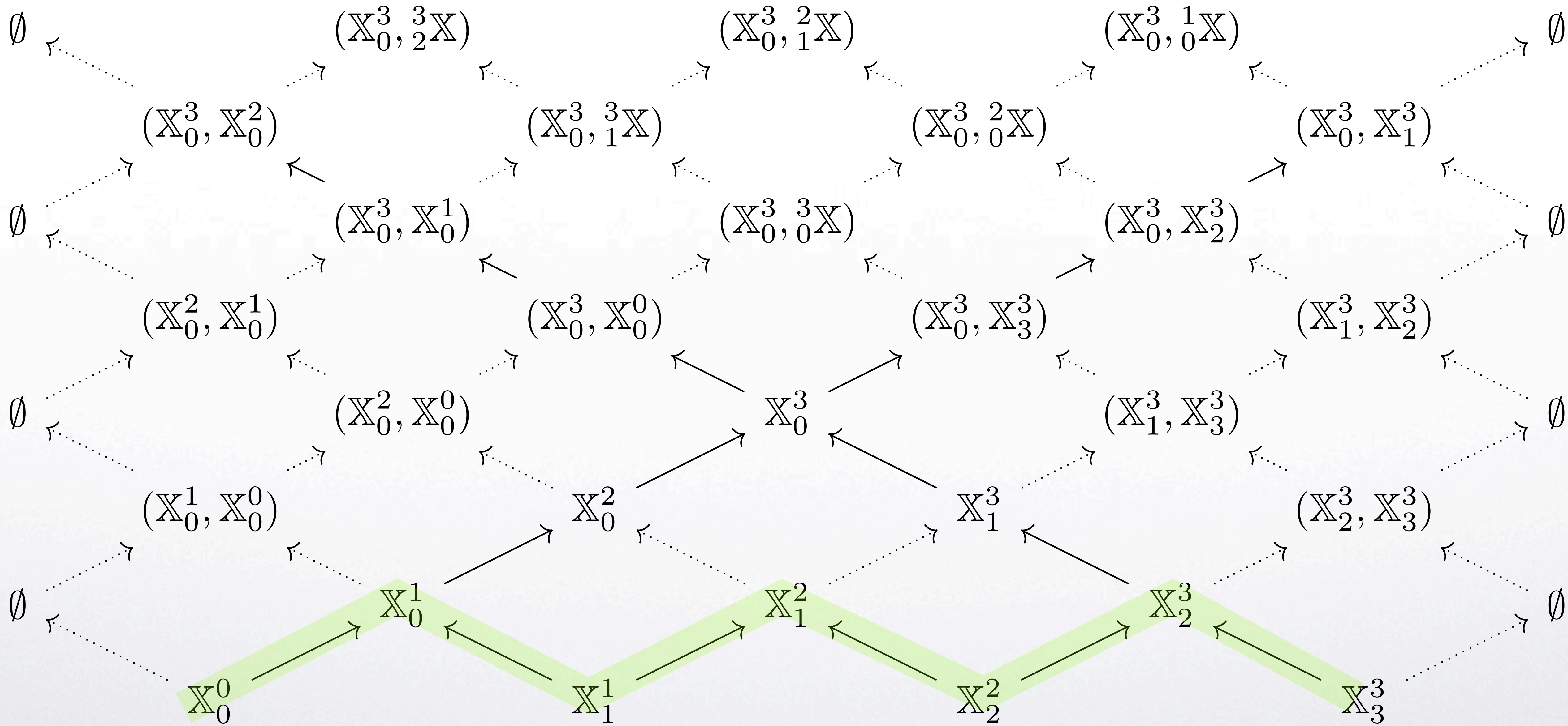


$\mathbb{X}_i^j = f^{-1}[i, j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

Vin de Silva
http://pages.pomona.edu/~vds04747/

A vast commutative diagram

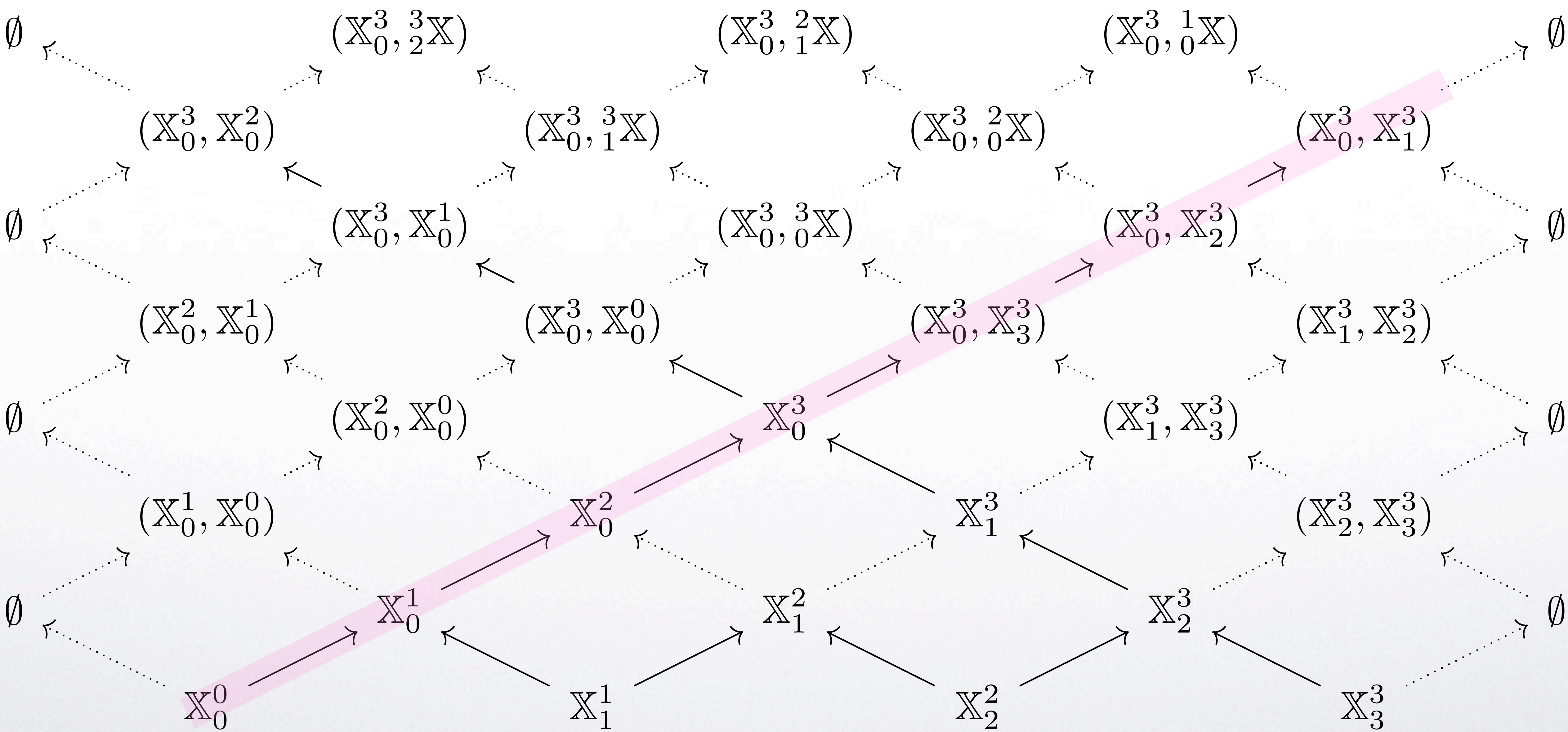


Levelset zigzag persistence

$\mathbb{X}_i^j = f^{-1}[i, j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

A vast commutative diagram

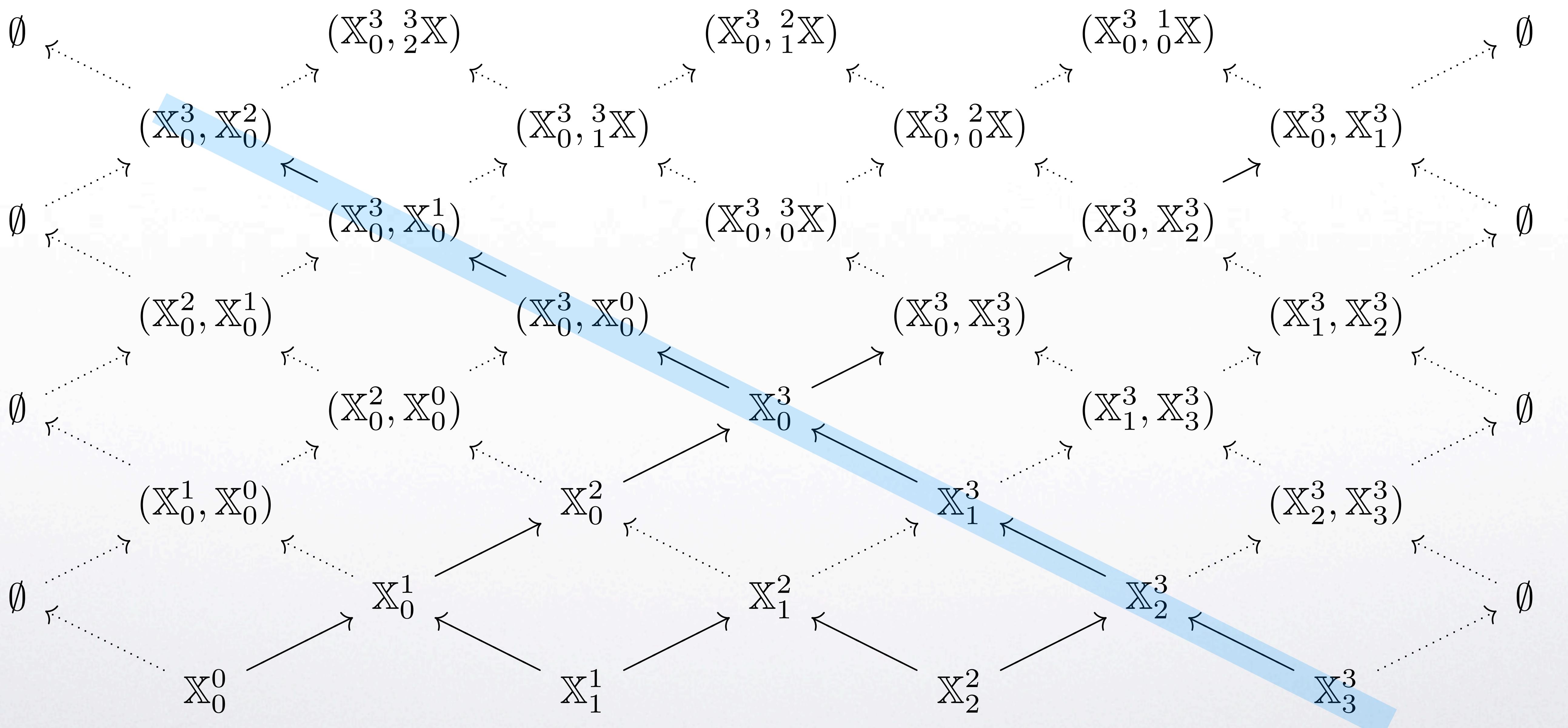


Extended persistence of (X,f)

$\mathbb{X}_i^j = f^{-1}[i,j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

A vast commutative diagram

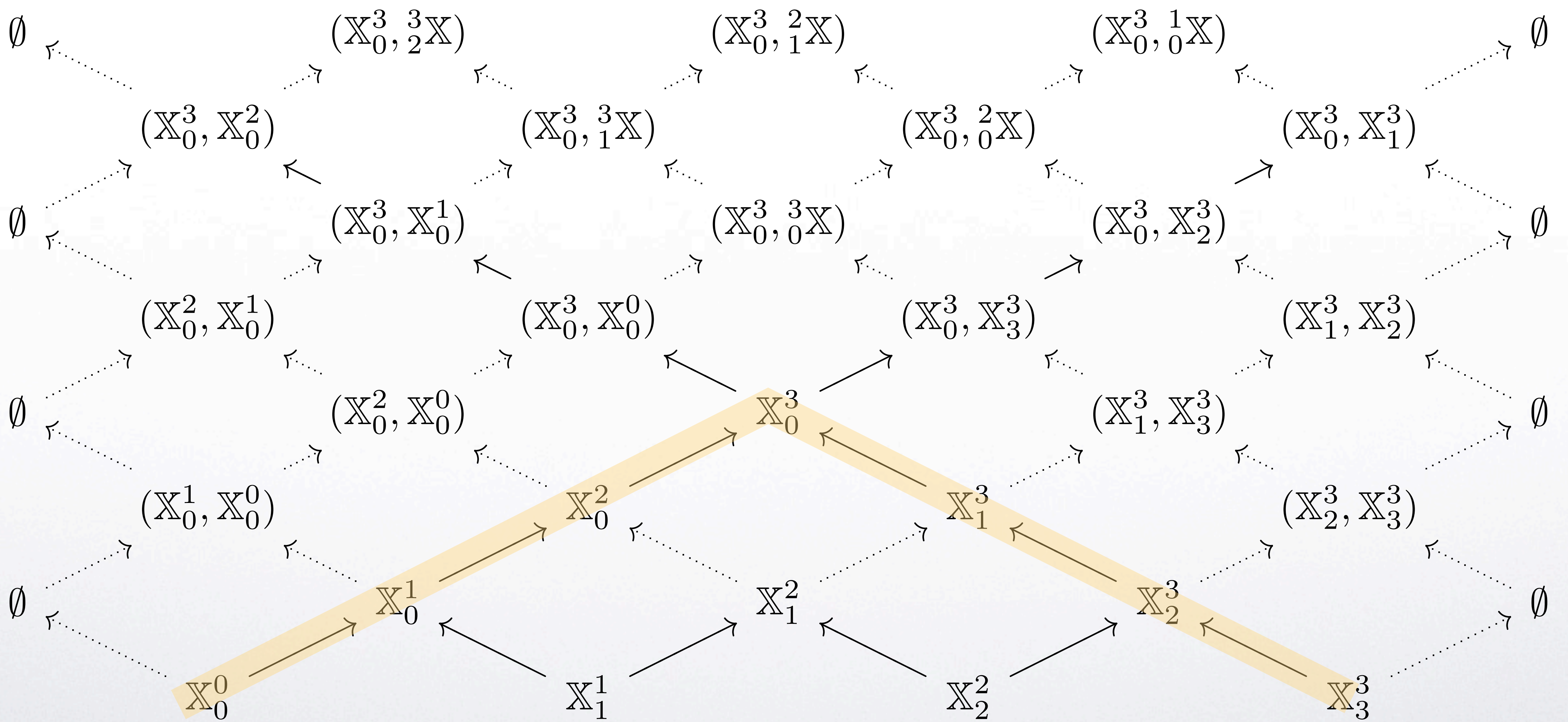


Extended persistence of (X,-f)

$\mathbb{X}_i^j = f^{-1}[i,j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

A vast commutative diagram

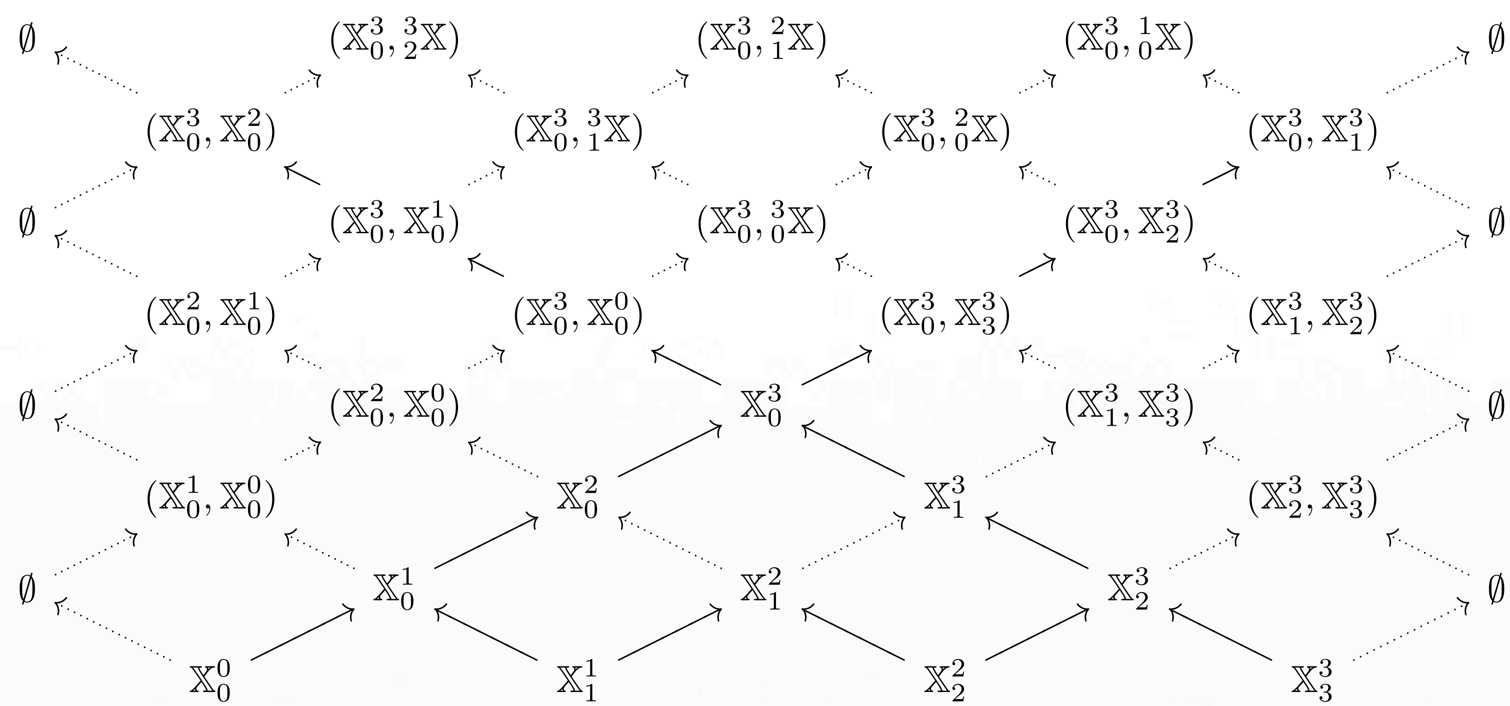


Up-down persistence of (X,f)

$\mathbb{X}_i^j = f^{-1}[i, j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

The pyramid theorem



- Every diamond is Mayer–Vietoris.
- Thus all monotone paths from left to right carry the same zigzag persistent information (rearranged, with dimension shifts).
- In particular, the following are equivalent:

Extended persistence of (X,f) | Extended persistence of (X,-f)

Levelset zigzag persistence | Up-down persistence of (X,f)

Thank you